Chapter No. 1

Complex Numbers

Exercise # 1.1

Q.1 Simplify the following:

(i) i^9

Sol:– $i^9 = i \cdot i^8$

$= i \cdot (i^2)^4$ $\boxed{\because i^2 = -1}$

$= i(-1)^4$

$= i(1)$

$\boxed{i^9 = i}$ Ans

ii) i^{17}

Sol:– $i^{17} = i \cdot i^{16}$

$= i \cdot (i^2)^8$ $\because i^2 = -1$

$= i(-1)^8$

$= i(1)$

$\boxed{i^{17} = i}$ Ans

iii) $(-i)^{19}$

Sol:– $(-i)^{19} = -i^{19}$

$= -i \cdot i^{18}$

$= -i \cdot (i^2)^9$ $\because i^2 = -1$

$(-i)^{19} = -i(-1)^9$

$= -i(-1)$

$\boxed{(-i)^{19} = i}$ Ans

(iv) $(-1)^{-\frac{21}{2}}$

Sol:– $(-1)^{-21/2} = (i^2)^{-\frac{21}{2}}$

$= i^{2 \times \frac{-21}{2}}$ $\because i^2 = -1$

$= i^{-21}$

$= i^{-22+1}$

$= i^{-22} \cdot i$

$= \frac{i}{i^{22}}$

$= \frac{i}{(i^2)^{11}}$

$= \frac{i}{(-1)^{11}}$

$= \frac{i}{-1}$

$\boxed{(-1)^{-\frac{21}{2}} = -i}$ Ans

Q.2 Prove that $\bar{z} = z$ iff z is real.

Sol:— Suppose that $\bar{z} = z$

we have to prove that z is real

Let $z = a + ib$ — (1

$\Rightarrow \bar{z} = a - ib$ — (2

from (1 & (2

$$a + ib = a - ib$$

$$\Rightarrow \cancel{a} + ib - \cancel{a} + ib = 0$$

$$2ib = 0$$

$$b = 0 \quad \text{but } 2i \neq 0$$

put $b = 0$ in eq. (1

$$z = a + i(0)$$

$$z = a$$

Hence z is real

Conversely Suppose that z is real,

we have to prove that $\bar{z} = z$

$$z = a \text{ — (i}$$

$$\Rightarrow \bar{z} = a \text{ — (ii}$$

from (i & (ii

$$\bar{z} = z$$

Hence $\bar{z} = z \Leftrightarrow z$ is real

Q.3 Prove that for $z \in C$

i) $\dfrac{z + \bar{z}}{2} = Re(z)$

Sol:— Let $z = a + ib$; $Re(z) = a$

$$\Rightarrow \bar{z} = a - ib$$

$$L.H.S = \frac{z + \bar{z}}{2}$$

$$= \frac{a + \cancel{ib} + a - \cancel{ib}}{2}$$

$$= \frac{\cancel{2}a}{\cancel{2}} = a$$

$$= Re(z) = R.H.S$$

Hence $\boxed{\dfrac{z + \bar{z}}{2} = Re(z)}$

ii) $\dfrac{z - \bar{z}}{2i} = Im(z)$

Sol:— Let $z = a + ib$; $Im(z) = b$

$$\Rightarrow \bar{z} = a - ib$$

$$L.H.S = \frac{z - \bar{z}}{2i}$$

$$= \frac{a+ib-(a-ib)}{2i}$$

$$= \frac{\cancel{a}+ib-\cancel{a}+ib}{2i}$$

$$= \frac{2bi}{2i} = b$$

$$= Im(z) = R.H.S$$

Hence $\boxed{\frac{z-\bar{z}}{2i} = Im(z)}$

iii) $|z|^2 = z \cdot \bar{z}$

Sol:-

Let $z = a+ib$

$L.H.S = |z|^2$

$|z| = \sqrt{(a)^2+(b)^2}$

$|z| = \sqrt{a^2+b^2}$

Taking Sq. on b/s

$\Rightarrow |z|^2 = a^2+b^2$ —— (1

$R.H.S = z \cdot \bar{z}$

$z = a+ib \Rightarrow \bar{z} = a-ib$

$z \cdot \bar{z} = (a+ib)(a-ib)$

$\because a^2-b^2 = (a-b)(a+b)$

$z \cdot \bar{z} = a^2-(ib)^2$

$= a^2 - i^2b^2$

$= a^2-(-1)b^2$

$z \cdot \bar{z} = a^2+b^2$ —— (2

From (1 & (2

Hence $\boxed{|z|^2 = z \cdot \bar{z}}$

iv) $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

Sol:-

Let $z = a+ib \Rightarrow \bar{z} = a-ib$

$L.H.S = \frac{1}{z} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib}$

$\frac{1}{z} = \frac{a-ib}{a^2-i^2b^2} = \frac{a-ib}{a^2-(-1)b^2}$

$\frac{1}{z} = \frac{a-ib}{a^2+b^2}$ —— (1

$R.H.S = \frac{\bar{z}}{|z|^2}$

$|z| = \sqrt{a^2+b^2}$

Taking Sq. on b/s

$|z|^2 = a^2+b^2$

$\Rightarrow \frac{\bar{z}}{|z|^2} = \frac{a-ib}{a^2+b^2}$ —— (2

From (1 & (2 Hence

$\boxed{\frac{1}{z} = \frac{\bar{z}}{|z|^2}}$

Q.3 Find the multiplicative Inverses of each of the following numbers:

i) $(-4, 7)$

Sol:- $(-4, 7)$

$a = -4$ & $b = 7$

$M.I \text{ of } (a,b) = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

putting the value of a & b

$M.I \text{ of } (-4,7) = \left(\frac{-4}{(-4)^2+7^2}, \frac{-7}{(-4)^2+(7)^2}\right)$

$= \left(\frac{-4}{16+49}, -\frac{7}{16+49}\right)$

$\boxed{M.I \text{ of } (-4,7) = \left(\frac{-4}{65}, \frac{-7}{65}\right)}$ Ans

ii) $(\sqrt{2}, -\sqrt{5})$

Sol:- $(\sqrt{2}, -\sqrt{5})$

$a = \sqrt{2}$, $b = -\sqrt{5}$

$M.I \text{ of } (a,b) = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

putting the value of a & b

$M.I \text{ of } (\sqrt{2}, -\sqrt{5}) = \left(\frac{\sqrt{2}}{(\sqrt{2})^2+(-\sqrt{5})^2}, \frac{-(-\sqrt{5})}{(\sqrt{2})^2+(-\sqrt{5})^2}\right)$

$= \left(\frac{\sqrt{2}}{2+5}, \frac{\sqrt{5}}{2+5}\right)$

$\boxed{M.I \text{ of } (\sqrt{2}, -\sqrt{5}) = \left(\frac{\sqrt{2}}{7}, \frac{\sqrt{5}}{7}\right)}$

iii) $(1, 0)$

Sol:- $(1, 0)$

$a = 1$, $b = 0$

$M.I \text{ of } (a,b) = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

putting the value of a & b

$M.I \text{ of } (1,0) = \left(\frac{1}{1^2+0^2}, \frac{-0}{1^2+0^2}\right)$

$= \left(\frac{1}{1+0}, \frac{0}{1+0}\right)$

$\boxed{M.I \text{ of } (1,0) = (1,0)}$ Ans

Q.5 Separate into real and imaginary parts

i) $\frac{2-7i}{4+5i}$

Sol:- Let $Z = \frac{2-7i}{4+5i}$

$Z = \frac{2-7i}{4+5i} \times \frac{4-5i}{4-5i}$

$Z = \frac{8-10i-28i+35i^2}{(4)^2-(5i)^2}$

$Z = \frac{8-38i+35(-1)}{16-25i^2}$

$Z = \frac{8-38i-35}{16-25(-1)}$

$Z = \frac{-27-38i}{16+25}$

$$Z = \frac{-27-38i}{41}$$

$$Z = -\frac{27}{41} - \frac{38}{41}i$$

$$\boxed{Re(z) = \frac{-27}{41} \ \& \ Im(z) = \frac{-38}{41}}$$ Ans

ii) $\frac{(-2+3i)^2}{1+i}$

Sol:- Let $Z = \frac{(-2+3i)^2}{1+i}$

$$Z = \frac{4+9i^2-12i}{1+i}$$

$$Z = \frac{4+9(-1)-12i}{1+i}$$

$$Z = \frac{4-9-12i}{1+i}$$

$$Z = \frac{-5-12i}{1+i}$$

$$Z = \frac{-5-12i}{1+i} \times \frac{1-i}{1-i}$$

$$Z = \frac{(-5-12i)(1-i)}{(1+i)(1-i)}$$

$$Z = \frac{-5+5i-12i+12i^2}{1^2-i^2}$$

$$Z = \frac{-5-7i+12(-1)}{1-(-1)}$$

$$Z = \frac{-5-7i-12}{1+1}$$

$$Z = \frac{-17-7i}{2}$$

$$Z = \frac{-17}{2} - \frac{7}{2}i$$

$$\boxed{Re(z) = -\frac{17}{2} \ \& \ Im(z) = \frac{-7}{2}}$$

iii) $\frac{i}{1+i}$

Sol:- Let $Z = \frac{i}{1+i}$

$$Z = \frac{i}{1+i} \times \frac{1-i}{1-i}$$

$$Z = \frac{i(1-i)}{(1+i)(1-i)}$$

$$Z = \frac{i-i^2}{1-i^2}$$

$$Z = \frac{i-(-1)}{1-(-1)}$$

$$Z = \frac{i+1}{1+1}$$

$$Z = \frac{1+i}{2}$$

$$Z = \frac{1}{2} + \frac{1}{2}i$$

$$\boxed{Im(z) = \frac{1}{2}, \ Re(z) = \frac{1}{2}}$$ Ans

(iv) $\dfrac{(4+3i)^2}{4-3i}$

Sol:- Let $Z = \dfrac{(4+3i)^2}{4-3i}$

$$Z = \frac{16+9i^2+24i}{4-3i}$$

$$Z = \frac{16+9(-1)+24i}{4-3i}$$

$$Z = \frac{16-9+24i}{4-3i}$$

$$Z = \frac{7+24i}{4-3i}$$

$$Z = \frac{7+24i}{4-3i} \times \frac{4+3i}{4+3i}$$

$$Z = \frac{28+21i+96i+72i^2}{(4)^2-(3i)^2}$$

$$Z = \frac{28+117i+72(-1)}{16-9i^2}$$

$$Z = \frac{28+117i-72}{16-9(-1)}$$

$$Z = \frac{-44+117i}{16+9}$$

$$Z = \frac{-44+117i}{25}$$

$$Z = -\frac{44}{25} + \frac{117}{25}i$$

$$\boxed{Re(z) = -\frac{44}{25} \ \& \ Im(z) = \frac{117}{25}}$$

Q.6 If $z_1 = 2+i$, $z_2 = 3-2i$, $z_3 = 1+3i$ then express $\dfrac{\bar{z}_1 \cdot \bar{z}_3}{z_2}$ in the form of $a+ib$

Sol:- Given that

$$z_1 = 2+i \Rightarrow \bar{z}_1 = 2-i$$

$$z_3 = 1+3i \Rightarrow \bar{z}_3 = 1-3i$$

$$z_2 = 3-2i$$

$$\bar{z}_1 \cdot \bar{z}_3 = (2-i)(1-3i)$$

$$= 2-6i-i+3i^2$$

$$= 2-7i+3(-1)$$

$$= 2-7i-3$$

$$\bar{z}_1 \cdot \bar{z}_3 = -1-7i$$

Now $\dfrac{\bar{z}_1 \cdot \bar{z}_3}{z_2} = \dfrac{-1-7i}{3-2i}$

$$= \frac{-1-7i}{3-2i} \times \frac{3+2i}{3+2i}$$

$$= \frac{-3-2i-21i-14i^2}{(3)^2-(2i)^2}$$

$$= \frac{-3-23i-14(-1)}{9-4i^2}$$

$$= \frac{-3-23i+14}{9-4(-1)}$$

$$= \frac{11-23i}{9+4} = \frac{11-23i}{13}$$

$$\boxed{\frac{\bar{z}_1 \cdot \bar{z}_3}{z_2} = \frac{11}{13} - \frac{23i}{13}}$$ Ans.

Q.7 If $z_1 = 2+7i$ and $z_2 = -5+3i$, the evaluate the following

i) $|2z_1 - 4z_2|$

Sol:- Given that

$z_1 = 2+7i$ and $z_2 = -5+3i$

$2z_1 - 4z_2 = 2(2+7i) - 4(-5+3i)$

$= 4 + 14i + 20 - 12i$

$2z_1 - 4z_2 = 24 + 2i$

Taking Modulus

$|2z_1 - 4z_2| = \sqrt{(24)^2 + (2)^2}$

$= \sqrt{576 + 4}$

$= \sqrt{580} = \sqrt{4 \times 145}$

$\boxed{|2z_1 - 4z_2| = 2\sqrt{145}}$ Ans

ii) $|3z_1 + 2\bar{z}_1|$

Sol:- Given that

$z_1 = 2+7i \Rightarrow \bar{z}_1 = 2-7i$

$3z_1 + 2\bar{z}_1 = 3(2+7i) + 2(2-7i)$

$= 6 + 21i + 4 - 14i$

$3z_1 + 2\bar{z}_1 = 10 + 7i$

Taking Modulus

$|3z_1 + 2\bar{z}_1| = \sqrt{(10)^2 + (7)^2}$

$= \sqrt{100 + 49} = \sqrt{149}$

$\boxed{|3z_1 + 2\bar{z}_1| = \sqrt{149}}$ Ans

iii) $|-7z_2 + 2\bar{z}_2|$

Sol:- Given that

$z_2 = -5+3i \Rightarrow \bar{z}_2 = -5-3i$

$-7z_2 + 2\bar{z}_2 = -7(-5+3i) + 2(-5-3i)$

$= 35 - 21i - 10 - 6i$

$= -27i$

Taking Modulus

$|-7z_2 + 2\bar{z}_2| = \sqrt{(25)^2 + (-27)^2}$

$= \sqrt{625 + 729}$

$\boxed{|-7z_2 + 2\bar{z}_2| = \sqrt{1354}}$ Ans

iv) $|(z_1 + z_2)^3|$

Sol:- Given that

$z_1 = 2+7i$ and $z_2 = -5+3i$

$z_1 + z_2 = 2 + 7i - 5 + 3i = -3 + 10i$

Taking cube

$(z_1 + z_2)^3 = (-3 + 10i)^3$

$= -27 + 1000i^3 + 3(-3)(10i)(-3 + 10i)$

$= -27 + 1000i^2 \cdot i - 90i(-3 + 10i)$

$= -27 - 1000i + 270i - 900i^2$

$\Rightarrow (z_1 + z_2)^3 = 873 - 730i$

$\Rightarrow |(z_1 + z_2)^3| = \sqrt{(873)^2 + (-730)^2}$

$|(z_1 + z_2)^3| = \sqrt{1295029} = 109\sqrt{109}$ Ans

Q.7 Show that $i^{n+1} + i^{n+2} + i^{n+3} + i^{n+4} = 0 \quad \forall n \in N$

Sol:- $i^{n+1} + i^{n+2} + i^{n+3} + i^{n+4} = 0$

$$L.H.S = i^{n+1} + i^{n+2} + i^{n+3} + i^{n+4}$$

$$= i^n \cdot i + i^n \cdot i^2 + i^n \cdot i^3 + i^n \cdot i^4$$

$$= i^n (i + i^2 + i^3 + i^4)$$

$$= i^n (i + i^2 + i \cdot i^2 + (i^2)^2)$$

$$= i^n (i + (-1) + i(-1) + (-1)^2)$$

$$= i^n (i - 1 - i + 1) = i^n (0)$$

$$= 0 = R.H.S$$

Hence $L.H.S = R.H.S$

Q.8 Find the least positive value of n, if $\left(\frac{1+i}{1-i}\right)^{2n} = 1$

Sol:- $L.H.S = \left(\frac{1+i}{1-i}\right)^{2n} = \left[\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right]^{2n}$

$$= \left[\frac{(1+i)^2}{1-i^2}\right]^{2n} = \left[\frac{1+i^2+2i}{1-(-1)}\right]^{2n} = \left(\frac{1-1+2i}{1+1}\right)^{2n}$$

$$= \left(\frac{2i}{2}\right)^{2n} = i^{2n}$$

put $n = 2$

$$= i^{2(2)} = i^4 = (i^2)^2 = (-1)^2$$

$= 1 =$ R.H.S

Thus The least value of $n = 2$.

Q.9 Show that, the value of i^n for $n \in N$ & $n > 4$ is i^r, Where r is the remainder when n is divided by 4.

Sol:- When n is divided by 4 we have divisor & get some quotient & remainder

we have to prove that

$$i^n = i^r \quad \text{for } n \in N \;\&\; n > 4$$

$\because$ Number/dividend = Divisor$\times q + r$

$$n = 4 \times q + r$$

$$\Rightarrow \quad i^n = i^{4q+r}$$

$$i^n = i^{4q} \cdot i^r$$

$$i^n = (i^2)^{2q} \cdot i^r$$

$$i^n = [(-1)^2]^q \cdot i^r$$

$$i^n = i^r (1)^q \qquad \because 1^q = 1$$

$$i^n = i^n$$

Hence Prove.