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Exercise # 1.4

Q.1 Find the three cube roots of:

i) 8

Sol:- Let x be the cube root 8

$\therefore x = (8)^{1/3}$

Taking cube on b/s

$x^3 = (8)^{\frac{1}{3} \cdot 3}$

$x^3 = 8 \Rightarrow x^3 - 8 = 0$

$x^3 - 2^3 = 0$

$\because (a^3 - b^3) = (a-b)(a^2+ab+b^2)$

$(x-2)(x^2+2x+2^2) = 0$

$(x-2)(x^2+2x+4) = 0$

$x-2 = 0 \wedge x^2+2x+4 = 0$

$\boxed{x = 2}$ using Q.F

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

put $a=1,\ b=2,\ c=4$

$x = \dfrac{-2 \pm \sqrt{2^2-4(1)(4)}}{2(1)}$

$x = \dfrac{-2 \pm \sqrt{4-16}}{2}$

$x = \dfrac{-2 \pm \sqrt{-12}}{2}$

$x = \dfrac{-2 \pm \sqrt{4 \times -3}}{2}$

$x = \dfrac{-2 \pm 2\sqrt{-3}}{2}$

$x = 2\left(\dfrac{-1 \pm \sqrt{-3}}{2}\right)$

$x = 2\left(\dfrac{-1-\sqrt{-3}}{2}\right) \wedge x = \dfrac{2(-1+\sqrt{-3})}{2}$

$\boxed{x = 2\omega}$ & $\boxed{x = 2\omega^2}$

Hence cube roots of 8 are

$2, 2\omega, 2\omega^2$

ii) -8

Sol:- Let x be the cube root -8

$\therefore x = (-8)^{1/3}$

Taking cube on b/s

$x^3 = (-8)^{\frac{1}{3} \cdot 3}$

$x^3 = -8 \Rightarrow x^3 + 8 = 0$

$x^3 + 2^3 = 0$

$\because a^3 + b^3 = (a+b)(a^2-ab+b^2)$

$(x+2)(x^2-2x+2^2) = 0$

$(x+2)(x^2-2x+4) = 0$

$x+2 = 0 \wedge x^2-2x+4 = 0$

$\boxed{x = -2}$ using Q.F

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

put $a=1,\ b=-2,\ c=4$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{4-16}}{2}$$

$$x = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm \sqrt{4 \times -3}}{2}$$

$$x = \frac{2 \pm 2\sqrt{-3}}{2} = 2\left(\frac{1 \pm \sqrt{-3}}{2}\right)$$

$$x = 2\left(\frac{1+\sqrt{-3}}{2}\right) \wedge x = 2\left(\frac{1-\sqrt{-3}}{2}\right)$$

$$x = -2\left(\frac{-1-\sqrt{-3}}{2}\right) \quad x = -2\left(\frac{-1+\sqrt{-3}}{2}\right)$$

$$\boxed{x = -2\omega} \qquad \boxed{x = -2\omega^2}$$

Hence, The three cube roots of -8 are, $-2, -2\omega, -2\omega^2$

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iii) -27

Sol:- Let x be the cube root of -27

$$\therefore x = (-27)^{1/3}$$

Taking cube on b/s

$$x^3 = (-27)^{\cancel{1/3} \cdot \cancel{3}}$$

$$x^3 = -27 \Rightarrow x^3 + 27 = 0$$

$$x^3 + 3^3 = 0$$

$$\because (a^3+b^3) = (a+b)(a^2-ab+b^2)$$

$$(x+3)(x^2-3x+3^2) = 0$$

$$(x+3)(x^2-3x+9) = 0$$

$$x+3 = 0 \wedge x^2-3x+9 = 0$$

$$\boxed{x = -3}$$

using Q.F

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

put $a = 1, b = -3, c = 9$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)}$$

$$x = \frac{3 \pm \sqrt{9-36}}{2}$$

$$x = \frac{3 \pm \sqrt{-27}}{2} = \frac{3 \pm \sqrt{9 \times -3}}{2}$$

$$x = \frac{3 \pm 3\sqrt{-3}}{2} = 3\left(\frac{1 \pm \sqrt{-3}}{2}\right)$$

$$x = 3\left(\frac{1+\sqrt{-3}}{2}\right) \wedge x = 3\left(\frac{1-\sqrt{-3}}{2}\right)$$

$$x = -3\left(\frac{-1-\sqrt{-3}}{2}\right) \wedge x = -3\left(\frac{-1+\sqrt{-3}}{2}\right)$$

$$\boxed{x = -3\omega^2} \qquad \boxed{x = -3\omega}$$

Hence, cube roots of -27 are, $-3, -3\omega, -3\omega^2$

(iv) 64

Sol:- Let x be the cube root of 64

$$\therefore x = (64)^{1/3}$$

Taking cube on b/s

$$x^3 = (64)^{\cancel{1/3} \cdot \cancel{3}}$$

$$x^3 = 64 \Rightarrow x^3 - 64 = 0$$

$$x^3 - 4^3 = 0$$

$$\because (a^3-b^3) = (a-b)(a^2+ab+b^2)$$

$$(x-4)(x^2+4x+4^2)=0$$
$$(x-4)(x^2+4x+16)=0$$
$$x-4=0 \wedge x^2+4x+16=0$$
$$\boxed{x=4}$$ using Q.F
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-4\pm\sqrt{4^2-4(1)(16)}}{2(1)}$$
$$x=\frac{-4\pm\sqrt{16-64}}{2}$$
$$x=\frac{-4\pm\sqrt{-48}}{2}$$
$$x=\frac{-4\pm\sqrt{16\times-3}}{2}$$
$$x=\frac{-4\pm4\sqrt{-3}}{2}=4\left(\frac{-1\pm\sqrt{-3}}{2}\right)$$
$$x=4\left(\frac{-1-\sqrt{-3}}{2}\right) \wedge x=4\left(\frac{-1+\sqrt{-3}}{2}\right)$$
$$\boxed{x=4\omega} \ \& \ \boxed{x=4\omega^2}$$
Hence, The three cube roots of 64 are, 4, 4ω, $4\omega^2$

(v) -125

Sol:- Let x be the cube root of -125
$$\therefore x=(-125)^{1/3}$$
Taking cube on b/s
$$x^3=(-125)^{\frac{1}{3}\cdot 3}$$
$$x^3=-125 \Rightarrow x^3+125=0$$
$$x^3+5^3=0$$
$$\because a^3+b^3=(a+b)(a^2-ab+b^2)$$
$$(x+5)(x^2-5x+5^2)=0$$
$$(x+5)(x^2-5x+25)=0$$
$$x+5=0 \wedge x^2-5x+25=0$$
$$\boxed{x=-5}$$ using Q.F
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
put $a=1$, $b=-5$, $c=25$
$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(25)}}{2(1)}$$
$$x=\frac{5\pm\sqrt{25-100}}{2}$$
$$x=\frac{5\pm\sqrt{-75}}{2}$$
$$x=\frac{5\pm\sqrt{25\times-3}}{2}$$
$$x=\frac{5\pm5\sqrt{-3}}{2}=5\left(\frac{5\pm\sqrt{-3}}{2}\right)$$
$$x=5\left(\frac{1\pm\sqrt{-3}}{2}\right) \wedge x=5\left(\frac{1-\sqrt{-3}}{2}\right)$$
$$x=-5\left(\frac{-1-\sqrt{-3}}{2}\right) \wedge x=-5\left(\frac{-1+\sqrt{-3}}{2}\right)$$
$$\boxed{x=-5\omega} \ \& \ \boxed{x=-5\omega^2}$$
cube roots of -625 are -5, -5ω, $-5\omega^2$

Q.2 Find the four fourth roots of 16, 81, 625. Also show that their sum is zero in each case.

Sol:- Let x be the fourth roots of 16 i.e

$$x = (16)^{1/4}$$

Taking power 4 on b/s

$$\Rightarrow x^4 = 16$$

$$x^4 - 16 = 0$$

$$x^4 - 2^4 = 0$$

$$(x^2)^2 - (2^2)^2 = 0$$

$\because a^2 - b^2 = (a-b)(a+b)$

$$(x^2 - 2^2)(x^2 + 2^2) = 0$$

$$x^2 - 2^2 = 0 \quad \& \quad x^2 + 2^2 = 0$$

$$x^2 = 2^2 \qquad x^2 = -2^2$$

$$\Rightarrow \boxed{x = \pm 2} \qquad \Rightarrow \boxed{x = \pm 2i}$$

Thus four fourth roots of 16 are ± 2 & $\pm 2i$

$$\text{Sum} = \cancel{2} - \cancel{2} + \cancel{2i} - \cancel{2i}$$

$$\Rightarrow \boxed{S = 0}$$

Let x be the four roots of 81 i.e $x = (81)^{1/4}$

Taking power 4 on b/s

$$\Rightarrow x^4 = 81$$

$$x^4 - 81 = 0$$

$$x^4 - 3^4 = 0$$

$$(x^2)^2 - (3^2)^2 = 0$$

$\because a^2 - b^2 = (a-b)(a+b)$

$$(x^2 - 3^2)(x^2 + 3^2) = 0$$

$$x^2 - 3^2 = 0 \quad \& \quad x^2 + 3^2 = 0$$

$$x^2 = 3^2 \quad \& \quad x^2 = -3^2$$

$$\Rightarrow \boxed{x = \pm 3} \qquad \Rightarrow \boxed{x = \pm 3i}$$

Thus, four fourth roots of 81 are ± 3 & $\pm 3i$

$$S = \cancel{3} - \cancel{3} + \cancel{3i} - \cancel{3i}$$

$$\boxed{S = 0}$$

Let x be the four fourth roots of 625 i.e

$$x = (625)^{1/4}$$

$\Rightarrow x^4 = 625$

$x^4 - 625 = 0$

$(x^2)^2 - (25)^2 = 0$

$(x^2)^2 - (5^2)^2 = 0$

$\because a^2 - b^2 = (a-b)(a+b)$

$[(x^2) - (5^2)][x^2 + 5^2] = 0$

$x^2 - 5^2 = 0$ & $x^2 + 5^2 = 0$

$x^2 = 5^2$ $\quad$ $x^2 = -5^2$

$\Rightarrow \boxed{x = \pm 5}$ $\quad \Rightarrow \boxed{x = \pm 5i}$

Thus, the four fourth roots of 625 are ± 5 & $\pm 5i$

$S = \cancel{5} - \cancel{5} + \cancel{5i} - \cancel{5i}$

$\boxed{S = 0}$

Q.3 If $1, \omega, \omega^2$ are the cube roots of unity, Show that $1 + \omega^n + \omega^{2n} = 3$ where n is the multiple of 3 respectively.

Sol:- Given that

$1, \omega$ & ω^2 are the cube roots of unity.

We have to prove that

$1 + \omega^n + \omega^{2n} = 3$

$L.H.S = 1 + \omega^n + \omega^{2n}$

Let $n = 3K$ be the multiple of 3 respectively.

$= 1 + \omega^{3K} + \omega^{2(3K)}$

$= 1 + \omega^{3K} + \omega^{6K}$

$= 1 + (\omega^3)^K + (\omega^6)^K$

$\because \omega^3 = \omega^6 = 1$

$= 1 + 1^K + 1^K$

$= 1 + 1 + 1 = 3 = R.H.S$

Hence, $L.H.S = R.H.S$

Q.4 Evaluate

(i) $\left(\frac{-1 + \sqrt{-3}}{2}\right)^7 + \left(\frac{-1 - \sqrt{-3}}{2}\right)^7$

Sol:- $= \left(\frac{-1 + \sqrt{-3}}{2}\right)^7 + \left(\frac{-1 - \sqrt{-3}}{2}\right)^7$

As $\omega = \frac{-1 + \sqrt{-3}}{2}$ & $\omega^2 = \frac{-1 - \sqrt{-3}}{2}$

$= \omega^7 + (\omega^2)^7$

$= \omega \cdot \omega^6 + \omega^{14}$

$= \omega(1) + \omega^{12} \cdot \omega^2$

$= \omega + \omega^2(1)$

$\Rightarrow \quad = \omega + \omega^2 = -1$

$$\boxed{\left(\frac{-1+\sqrt{-3}}{2}\right)^7 + \left(\frac{-1-\sqrt{-3}}{2}\right)^7 = -1}$$ Ans

ii) $(-1+\sqrt{-3})^5 + (-1-\sqrt{-3})^5$

Sol: $(-1+\sqrt{-3})^5 + (-1-\sqrt{-3})^5$

$\omega = \frac{-1+\sqrt{-3}}{2} \Rightarrow 2\omega = -1+\sqrt{-3}$

$\omega^2 = \frac{-1-\sqrt{-3}}{2} \Rightarrow 2\omega^2 = -1-\sqrt{-3}$

$= (2\omega)^5 + (2\omega^2)^5 = 32\omega^5 + 32\omega^{10} = 32(\omega^5+\omega^{10})$

$= 32(\omega^2\cdot\omega^3 + \omega\cdot\omega^9) = 32(\omega^2\cdot 1 + \omega\cdot 1) = 32(\omega+\omega^2)$

$= 32(-1) = -32$

$$\Rightarrow \boxed{(-1+\sqrt{-3})^5 + (-1-\sqrt{-3})^5 = -32}$$ Ans

Q.5 Show that

$(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16})\ldots$ to $2n$ factors $= 2^{2n}$.

Sol:- $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16})\ldots$ to $2n$ factors $= 2^{2n}$

L.H.S $= (1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16})\ldots$ to $2n$ factors

$= [(1-\omega+\omega^2)(1-\omega^2+\omega^4)][(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16})]\ldots$ to $2n$ factors

$= [(1-\omega+\omega^2)(1-\omega^2+\omega^3\cdot\omega)][(1-\omega\cdot\omega^3+\omega^2\cdot\omega^6)(1-\omega^2\cdot\omega^6+\omega\cdot\omega^{15})]\ldots$ to $2n$ factors

As $\omega^3 = \omega^6 = \omega^{15} = 1$ we get

$$= [(1+\omega^2-\omega)(1+\omega-\omega^2)][(1+\omega^2-\omega)(1+\omega-\omega^2)] \ldots \text{ 2n factors}$$

As $1+\omega^2 = -\omega$ & $1+\omega = -\omega^2$

$$= [(-\omega-\omega)(-\omega^2-\omega^2)][(-\omega-\omega)(-\omega^2-\omega^2)] \ldots \text{ 2n factors}$$

$$= [(-2\omega)(-2\omega^2)][(-2\omega)(-2\omega^2)] \ldots \text{ 2n factors}$$

$$= (4\omega^3)(4\omega^3)(4\omega^3) \ldots \text{ n factors}$$

$$= (4)(4)(4) \ldots \text{ n factors}$$

$$= 4^n = (2^2)^n = 2^{2n} = R.H.S$$

Hence $L.H.S = R.H.S$

Q.6 Prove that $\left(\frac{i+\sqrt{3}}{2}\right)^8 + \left(\frac{i-\sqrt{3}}{2}\right)^8 = -1$

Sol:- $\left(\frac{i+\sqrt{3}}{2}\right)^8 + \left(\frac{i-\sqrt{3}}{2}\right)^8 = -1$

$$L.H.S = \left(\frac{i+\sqrt{3}}{2}\right)^8 + \left(\frac{i-\sqrt{3}}{2}\right)^8$$

$$= \left[\frac{i(i+\sqrt{3})}{2i}\right]^8 + \left[\frac{i(i-\sqrt{3})}{2i}\right]^8$$

$$= \left[\frac{i^2+\sqrt{3}i}{2i}\right]^8 + \left[\frac{i^2-\sqrt{3}i}{2i}\right]^8 \quad \because i^2 = -1$$

$$= \left(\frac{-1+\sqrt{3}i}{2i}\right)^8 + \left(\frac{-1-\sqrt{3}i}{2i}\right)^8$$

$$= \frac{1}{i^8}\left[\frac{-1+\sqrt{3}i}{2}\right]^8 + \frac{1}{i^8}\left[\frac{-1-\sqrt{3}i}{2}\right]^8$$

As $\omega = \dfrac{-1+\sqrt{3}i}{2}$ & $\omega^2 = \dfrac{-1-\sqrt{3}i}{2}$

$$= \frac{1}{(i^2)^4}(\omega)^8 + \frac{1}{(i^2)^4}(\omega^2)^8 = \frac{1}{(-1)^4}\omega^8 + \frac{1}{(-1)^4}\omega^{16}$$

$$= \frac{1}{1}\omega^8 + \frac{1}{1}\omega^{16} = \omega^8 + \omega^{16} = \omega^2 \cdot \omega^6 + \omega \cdot \omega^{15}$$

$$= 1 \cdot \omega^2 + \omega \cdot 1 = \omega^2 + \omega = -1$$

Hence L.H.S = R.H.S

Q.7. Evaluate $\sum_{k=0}^{5} \omega^{2k}$, where ω is an imaginary cube root of unity.

Sol:-

$$\sum_{k=0}^{5} \omega^{2k} = \omega^{2(0)} + \omega^{2(1)} + \omega^{2(2)} + \omega^{2(3)} + \omega^{2(4)} + \omega^{2(5)}$$

$$= \omega^0 + \omega^2 + \omega^4 + \omega^6 + \omega^8 + \omega^{10}$$

$$= 1 + \omega^2 + \omega \cdot \omega^3 + 1 + \omega^2 \cdot \omega^6 + \omega \cdot \omega^9$$

As $\omega^3 = \omega^6 = \omega^9 = 1$

$$\sum_{k=0}^{5} \omega^{2k} = 1 + \omega^2 + \omega + 1 + \omega^2 + \omega$$

$$= 2 + 2\omega^2 + 2\omega = 2 + 2(\omega + \omega^2)$$

$$= 2 + 2(-1) = \cancel{2} - \cancel{2}$$

$$\sum_{k=0}^{5} \omega^{2k} = 0 \quad \text{Ans}$$

Q.8 If ω is an imaginary cube roots of unity, Prove that $\dfrac{a+b\omega^2+c\omega}{a\omega^2+b\omega+c} = \omega$

Sol:- $\dfrac{a+b\omega^2+c\omega}{a\omega^2+b\omega+c} = \omega$

$$L.H.S = \frac{a+b\omega^2+c\omega}{a\omega^2+b\omega+c} \quad \text{xing \& } \div\text{ing by } \omega$$

$$= \frac{(a+b\omega^2+c\omega)\omega}{(a\omega^2+b\omega+c)\omega} = \frac{\omega(a+b\omega^2+c\omega)}{a\omega^3+b\omega^2+c\omega} \quad \because \omega^3 = 1$$

$$= \frac{\omega\cancel{(a+b\omega^2+c\omega)}}{\cancel{a+b\omega^2+c\omega}} = \omega = R.H.S$$

Hence $L.H.S = R.H.S$

Q.9 If ω is a cube root of unity, Prove that

$$\frac{a\omega^{12}+b\omega^{17}+c\omega^{19}}{a\omega^{14}+b\omega^{22}+c\omega^{30}} = \omega$$

Sol:- $L.H.S = \dfrac{a\omega^{12}+b\omega^{17}+c\omega^{19}}{a\omega^{14}+b\omega^{22}+c\omega^{30}} = \dfrac{a\omega^{12}+b\omega^{15}\cdot\omega^2+c\omega\cdot\omega^{18}}{a\omega^2\cdot\omega^{12}+b\omega^{21}\cdot\omega+c\omega^{30}}$

$\because \omega^{12} = \omega^{15} = \omega^{18} = \omega^{21} = \omega^{30} = 1$

$$= \frac{a+b\omega^2+c\omega}{a\omega^2+b\omega+c}, \quad \text{xing \& } \div\text{ing by } \omega$$

$$= \frac{\omega(a+b\omega^2+c\omega)}{\omega(a\omega^2+b\omega+c)} = \frac{\omega(a+b\omega^2+c\omega)}{a\omega^3+b\omega^2+c\omega} \quad \because \omega^3 = 1$$

$$= \frac{\omega\cancel{(a+b\omega^2+c\omega)}}{\cancel{a+b\omega^2+c\omega}} = \omega = R.H.S$$

Hence $L.H.S = R.H.S$