Chapter No. #02

Functions and Graphs

Exercise # 2.1

Q.1 Given that

(a) $f(x) = x^2 - 1$ —— (1

(i) $f(-3) = ?$

Replace x by -3 in (1

$f(-3) = (-3)^2 - 1$

$= 9 - 1$

$\boxed{f(-3) = 8}$ Ans

ii) $f(0) = ?$

Replace x by 0 in (1

$f(0) = 0^2 - 1$

$\boxed{f(0) = -1}$ Ans

iii) $f(x-2)$

Replace x by $x-2$ in (1

$f(x-2) = (x-2)^2 - 1$

$= x^2 + 4 - 4x - 1$

$\boxed{f(x-2) = x^2 - 4x + 3}$ Ans

iv) $f(x^2+3)$

Replace x by x^2+3 in (1

$f(x^2+3) = (x^2+3)^2 - 1$

$= x^4 + 9 + 6x^2 - 1$

$\boxed{f(x^2+3) = x^4 + 6x^2 + 8}$ Ans

(b) $f(x) = \sqrt{2x+3}$ —— (1

i) $f(-3) = ?$

Replace x by -3

$f(-3) = \sqrt{2(-3)+3}$

$f(-3) = \sqrt{-6+3}$

$\boxed{f(-3) = \sqrt{-3}}$ Ans

ii) $f(0) = ?$

Replace x by 0 in (1

$f(0) = \sqrt{2(0)+3} = \sqrt{0+3}$

$\boxed{f(0) = \sqrt{3}}$ Ans

iii) $f(x-2) = ?$

Replace x by $x-2$ in (1

$$f(x-2) = \sqrt{2(x-2)+3}$$

$$= \sqrt{2x-4+3}$$

$$\boxed{f(x-2) = \sqrt{2x-1}}$$ Ans

iv) $f(x^2+3) = ?$

Replace x by x^2+3 in (1

$$f(x^2+3) = \sqrt{2(x^2+3)+3}$$

$$= \sqrt{2x^2+6+3}$$

$$\boxed{f(x^2+3) = 2x^2+9}$$ Ans

Q.2 Find $\dfrac{f(a+h)-f(a)}{h}$ and Simplify where

i) $f(x) = 4x+7$ —— (1

Replace x by $(a+h)$ in (1

$$f(a+h) = 4(a+h)+7$$

$$\boxed{f(a+h) = 4a+4h+7}$$

Replace x by a in (1

$$\boxed{f(a) = 4a+7}$$

Now $\dfrac{f(a+h)-f(a)}{h} = \dfrac{4a+4h+7-(4a+7)}{h}$

$$= \frac{\cancel{4a}+4h+\cancel{7}-\cancel{4a}-\cancel{7}}{h}$$

$$= \frac{4\cancel{h}}{\cancel{h}}$$

$$\boxed{\frac{f(a+h)-f(a)}{h} = 4}$$ Ans

ii) $f(x) = \sin x$ —— (1

Replace x by $(a+h)$ in eq. (1

$f(a+h) = \sin(a+h)$

Replace x by a in eq. (1

$f(a) = \sin(a)$

Now $\dfrac{f(a+h) - f(a)}{h} = \dfrac{\sin(a+h) - \sin(a)}{h}$

we know that

$\sin(P) - \sin(Q) = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

$\dfrac{f(a+h) - f(a)}{h} = \dfrac{1}{h} 2\cos\left[\dfrac{a+h+a}{2}\right]\sin\left[\dfrac{a+h-a}{2}\right]$

$= \dfrac{2}{h}\cos\left(\dfrac{2a+h}{2}\right)\sin\left(\dfrac{h}{2}\right)$

$$\boxed{\frac{f(a+h) - f(a)}{h} = \frac{2}{h}\cos\left(a + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}$$ Ans

iii) $f(x) = x^3 + x^2 - 1$ —— (1

Replace x by $(a+h)$ in eq. (1

$f(a+h) = (a+h)^3 + (a+h)^2 - 1$

$f(a+h) = a^3 + h^3 + 3a^2h + 3ah^2 + a^2 + h^2 + 2ah - 1$

Replace x by a in eq. (1

$$f(a) = a^3 + a^2 - 1$$

Now $\dfrac{f(a+h)-f(a)}{h} = \dfrac{1}{h}\left[a^3+h^3+3a^2h+3ah^2+a^2+h^2+2ah-1-(a^3+a^2-1)\right]$

$$= \frac{1}{h}\left[\cancel{a^3}+h^3+3a^2h+3ah^2+\cancel{a^2}+h^2+2ah\cancel{-1}\cancel{-a^3}\cancel{-a^2}\cancel{+1}\right]$$

$$= \frac{1}{h}\left[h^3+3a^2h+3ah^2+h^2+2ah\right]$$

$$= \frac{1}{\cancel{h}} \times \cancel{h}\left[h^2+3a^2+3ah+h+2a\right]$$

$$\boxed{\frac{f(a+h)-f(a)}{h} = h^2+3ah+h+3a^2+2a}$$ Ans

iv) $f(x) = \tan x$ ——— (1)

Replace x by $(a+h)$

$$f(a+h) = \tan(a+h)$$

Replace x by a

$$f(a) = \tan(a)$$

Now $\dfrac{f(a+h)-f(a)}{h} = \dfrac{1}{h}\left[\tan(a+h) - \tan(a)\right]$

$$= \frac{1}{h}\left[\frac{\sin(a+h)}{\cos(a+h)} - \frac{\sin(a)}{\cos(a)}\right]$$

$$= \frac{1}{h}\left[\frac{\sin(a+h)\cos(a) - \cos(a+h)\sin(a)}{\cos(a+h)\cos a}\right]$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

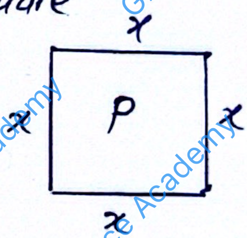
$$\frac{f(a+h)-f(a)}{h} = \frac{1}{h}\left[\frac{\sin(a+h-a)}{\cos(a)\cos(a+h)}\right]$$

$$\boxed{\frac{f(a+h)-f(a)}{h} = \frac{\sin(h)}{h\cos(a)\cos(a+h)}}$$ Ans

Q.3 Express the following:

(a) The Area A of Square as a function a function of its perimeter P.

Sol:- Let x be the sides of square also P is the perimeter of the square



Area of Square = $l \times w$

$$A = x \cdot x$$

$$A = x^2 \quad \text{———} \quad (1$$

Perimeter = Sum of all sides

$$P = x + x + x + x$$

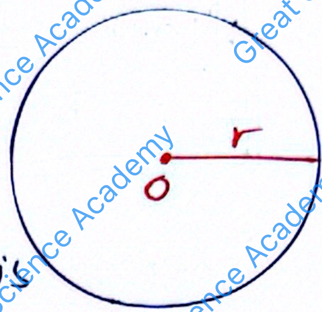
$$P = 4x \Rightarrow x = \frac{P}{4} \quad \text{put in } (1$$

$$A = \left(\frac{P}{4}\right)^2 \Rightarrow A = \frac{P^2}{16}$$

$$\boxed{A = \frac{1}{16}P^2}$$ which is req.

(b) The Circumference C of a circle as a function of its area A.

Let r be the radius & C be the Circumference of the Circle



Circumference of the circle is

$$C = 2\pi r \text{ ——— } (1$$

Area of the circle is $A = \pi r^2$

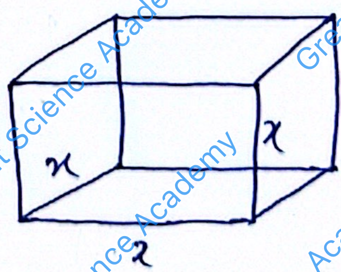
$$r^2 = \frac{A}{\pi} \Rightarrow r = \frac{\sqrt{A}}{\sqrt{\pi}} \quad \text{put in } (1$$

$$C = 2\pi \cdot \frac{\sqrt{A}}{\sqrt{\pi}} = 2(\sqrt{\pi})^2 \cdot \frac{\sqrt{A}}{\cancel{\sqrt{\pi}}} = 2\sqrt{\pi} \cdot \sqrt{A}$$

$$\boxed{C = 2\sqrt{\pi A}}$$ which is Req.

(c) The surface area S of cube as a function of its volume V.

Let x be the sides of cube



$$\text{Surface Area} = S = 6x^2 \text{ ——— } (1$$

$$V = l \times w \times h = x \cdot x \cdot x$$

$$V = x^3 \Rightarrow x = V^{1/3} \quad \text{put in } (1$$

$$S = 6(V^{1/3})^2$$

$$\boxed{S = 6V^{2/3}}$$ Which is req.

Q.4 Find the domain and the range of the function "g" defined below:

(i) $g(x) = 5 - x$

Sol:- $g(x) = 5 - x$

As $g(x)$ is a linear function so

$$D_{g(x)} = \mathbb{R} \text{ or } (-\infty, +\infty)$$

$$R_{g(x)} = \mathbb{R}^{+} \text{ or } (-\infty + \infty)$$

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ii) $g(x) = \sqrt{x+2}$

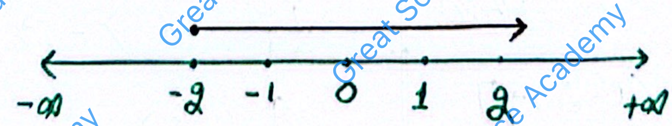
Sol:- $g(x) = \sqrt{x+2}$

For Domain:

put $x + 2 \geq 0$

$x \geq -2$

Real Line:



$$D_{g(x)} = [-2, +\infty)$$

As $g(x)$ is a radical function so

$$R_{g(x)} = [0, +\infty)$$

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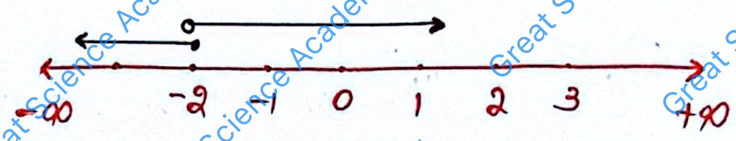
iv)

$$g(x) = \begin{cases} 6x+7 \ ; & x \le -2 \\ 4-3x \ ; & x > -2 \end{cases}$$

Sol:-

$$g(x) = \begin{cases} 6x+7 \ ; & x \le -2 \\ 4-3x \ ; & x > -2 \end{cases}$$

Real Line



For Domain

$$D_{g(x)} = (-\infty, -2] \cup (-2, +\infty)$$

$$= (-\infty, +\infty)$$

$$= \mathbb{R}$$

For Range

$g_1(x) = 6x+7$

If $x \le -2$ then

$g_1(-2) = 6(-2)+7$

$= -12+7$

$g_1(-2) = -5$

$g_2(x) = 4-3x$

If $x > -2$ then

$g_2(-2) = 4-3(-2)$

$= 4+6$

$g_2(-2) = 10$

$$R_{g(x)} = (-\infty, -5] \cup (10, +\infty)$$

iv) $g(x) = |x-5|$

Sol:- $g(x) = |x-5|$

As $g(x)$ is a absolute function therefore

$$D_{g(x)} = \mathbb{R} \text{ or } (-\infty, +\infty)$$

$$R_{g(x)} = \mathbb{R}^+ \text{ or } [0, +\infty)$$

Q.5 Given $f(x) = x^3 - ax^2 + bx + 1$. If $f(2) = -3$ and $f(-1) = 0$. Find the values of "a" and "b"

Sol:- Given that

$$f(x) = x^3 - ax^2 + bx + 1$$

$f(x) = x^3 - ax^2 + bx + 1$

put $x = 2$

$f(2) = (2)^3 - a(2)^2 + b(2) + 1$

$f(2) = 8 - 4a + 2b + 1$

$f(2) = 9 - 4a + 2b$

Given $f(2) = -3$

$-3 = 9 - 4a + 2b$

$\Rightarrow 4a - 2b = 9 + 3$

$4a - 2b = 12$

$\Rightarrow 2a - b = 6$ —— (1

$f(x) = x^3 - ax^2 + bx + 1$

put $x = -1$

$f(-1) = (-1)^3 - a(-1)^2 + b(-1) + 1$

$f(-1) = \cancel{-1} - a(1) + b(-1) + \cancel{1}$

$f(-1) = -a - b$

Given $f(-1) = 0$

$0 = -a - b$

$a = -b$ —— (2

put in (1

$2(-b) - b = 6$

$-2b-b=6 \Rightarrow -3b=6 \Rightarrow b=-6/3$

$\boxed{b=-2}$ put in eq. (2 we get

$a=-(-2) \Rightarrow \boxed{a=2}$

Q.6 Find the Domain and range of $g(x)=\frac{x+2}{3-x}$

Sol:- $g(x)=\frac{x+2}{3-x}$

For Domain

put $3-x\neq 0$

$3\neq x$

$D_{g(x)} = \mathbb{R}-\{3\}$ or $(-\infty,3)\cup(3,+\infty)$

For Range

As $g(x)$ in the form of $\frac{\text{Linear}}{\text{Linear}}$ therefore

$R_{g(x)} = \mathbb{R}-\{-1\}$ or $(-\infty,-1)\cup(-1,+\infty)$

Q.7 A Stone falls from a height of 60m on the ground, the height h after x Seconds is approximately given by $h(x)=40-10x^2$.

(i) What is the height of Stone when

(a) $x = 1$ sec?

Sol:- Given that

put $x = 1$ in $h(x) = 40 - 10x^2$

$h(1) = 40 - 10(1) = 40 - 10$

$\boxed{h(1) = 30 \text{ sec}}$ Ans

(b) $x = 1.5$ sec

put $x = 1.5$ in $h(x) = 40 - 10x^2$

$h(1.5) = 40 - 10(1.5)^2 = 40 - 10(2.25) = 40 - 22.5$

$\boxed{h(1.5) = 17.5 \text{ sec}}$ Ans

(c) $x = 1.7$ sec

put $x = 1.7$ in $h(x) = 40 - 10x^2$

$h(1.7) = 40 - 10(1.7)^2 = 40 - 10(2.89) = 40 - 28.9$

$\boxed{h(1.7) = 11.1 \text{ sec}}$ Ans

ii) When does the Stone Strike the ground?

$\because$ When Stone Strike the ground then $h(x) = 0$

$\Rightarrow \quad 40 - 10x^2 = 0 \Rightarrow 40 = 10x^2 \Rightarrow x^2 = 4$

$x = \pm 2 \quad \Rightarrow \quad \boxed{x = 2 \text{ sec}}$

Q. 8 Consider the function $f(x) = 3x - 5$

(i) Determine the domain & Range of $f(x)$

Given that $f(x) = 3x - 5$

As $f(x)$ is a linear function therefore

$$D_{f(x)} = \mathbb{R} = (-\infty, +\infty)$$

$$R_{f(x)} = \mathbb{R} = (-\infty, +\infty)$$

ii) Is the function f one-to-one? Justify the answer.

Ans: Yes the function f is one-to-one.

Justification

Suppose that $f(x_1) = f(x_2)$ for one-to-one

we want to prove that $x_1 = x_2$ & $x_1, x_2 \in \mathbb{R}$

$f(x_1) = 3x_1 - 5$ & $f(x_2) = 3x_2 - 5$

$$\Rightarrow 3x_1 - 5 = 3x_2 - 5$$

$$\Rightarrow 3x_1 = 3x_2$$

$$\Rightarrow x_1 = x_2$$

Hence, f is one to one function.

iii) Is the function f onto if the Co-domain is all real numbers? Explain.

Ans: Yes the function f onto

For onto we have to show that

$f(x) = y$

Given that $f(x) = 3x - 5$ —— (1)

let $y = 3x - 5 \Rightarrow 3x = y + 5$

$x = \frac{y+5}{3}$ put in (1) we get

$f(x) = \cancel{3}\left(\frac{y+5}{\cancel{3}}\right) - 5 = y + \cancel{5} - \cancel{5}$

$f(x) = y \Rightarrow f$ is onto.

Q.9 Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = \frac{2x-3}{x+1}$

i) Find the domain and range of $f(x)$

Given function

$$f(x) = \frac{2x-3}{x+1}$$

For Domain

put $x + 1 \neq 0$

$x \neq -1$

$D_{f(x)} = \mathbb{R} - \{-1\}$ or $(-\infty, -1) \cup (-1, +\infty)$

For Range

As $f(x)$ in the form of $\frac{\text{Linear}}{\text{Linear}}$ so

$R_{f(x)} = \mathbb{R} - \{2\}$ or $(-\infty, 2) \cup (2, +\infty)$

ii) Determine whether $f(x)$ is onto

For onto we have to show that

$f(x) = y$

Let $y = \frac{2x-3}{x+1} \Rightarrow y(x+1) = 2x - 3$

$$yx + y = 2x - 3 \Rightarrow yx - 2x = -3 - y$$

$$x(y-2) = -3-y \Rightarrow x = \frac{-3-y}{y-2}$$

put $x = \frac{-3-y}{y-2}$ in $f(x) = \frac{2x-3}{x+1}$

$$f(x) = \frac{2\left(\frac{-3-y}{y-2}\right) - 3}{\frac{-3-y}{y-2} + 1} = \frac{\frac{\cancel{-6} - 2y - 3y \cancel{+6}}{\cancel{y-2}}}{\frac{-3 \cancel{-y} \cancel{+y} - 2}{\cancel{y-2}}}$$

$$f(x) = \frac{-5y}{-5} \Rightarrow f(x) = y$$

Hence f is onto.

iii) Prove that $f(x)$ is one-to-one.

$$f(x) = \frac{2x-3}{x+1}$$

Suppose that $f(x_1) = f(x_2)$ for one-to-one

we have to prove that $x_1 = x_2$, $x_1, x_2 \in \mathbb{R}$

$$f(x_1) = \frac{2x_1 - 3}{x_1 + 1}, \quad f(x_2) = \frac{2x_2 - 3}{x_2 + 1}$$

$$\Rightarrow \frac{2x_1 - 3}{x_1 + 1} = \frac{2x_2 - 3}{x_2 + 1}$$

$$\Rightarrow (2x_1 - 3)(x_2 + 1) = (2x_2 - 3)(x_1 + 1)$$

$$\cancel{2x_1x_2} + 2x_1 - 3x_2 \cancel{-3} = \cancel{2x_1x_2} + 2x_2 \cancel{-3} - 3x_1$$

$$\Rightarrow 2x_1 + 3x_1 = 2x_2 + 3x_2 \Rightarrow 5x_1 = 5x_2$$

$\Rightarrow x_1 = x_2$ Hence f is one-to-one.

Q.10 Consider the function $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by $f(x) = e^{-x}$. Show that $f(x)$ is bijective.

Sol:- Given that

$f: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ defined by $f(x) = e^{-x}$

We have to prove that $f(x)$ is bijective.

For bijective we prove that the function is one-to-one & onto.

For one-to-one

Suppose that $f(x_1) = f(x_2)$, we have to prove that $x_1 = x_2$, $x_1, x_2 \in \mathbb{R}$

$f(x_1) = e^{-x_1}$ & $f(x_2) = e^{-x_2}$

$e^{-x_1} = e^{-x_2} \Rightarrow -x_1 = -x_2$

$\Rightarrow x_1 = x_2$ Hence f is one-to-one

For onto

We have to prove that $f(x) = y$

$f(x) = e^{-x}$ ——— 1

Let $y = e^{-x}$

Taking ln on b/s

$\ln y = \ln e^{-x} \Rightarrow \ln y = -x \ln e$ $\quad \because \ln e = 1$

$\ln y = -x$ put in (1

$$f(x) = e^{\ln y} \Rightarrow f(x) = y$$

Hence $f(x)$ is onto

Thus the function $f(x)$ is bijective.

Q.11 Let $g: \mathbb{R} \to \mathbb{R}$ is given by $g(x) = x^3 - 3x$. Determine if $g(x)$ is injective and/or surjective.

For Injective Given that $g(x) = x^3 - 3x$

Suppose that $g(x_1) = g(x_2)$ we have to prove that $x_1 = x_2$, $x_1, x_2 \in \mathbb{R}$

$$g(x_1) = x_1^3 - 3x_1 \quad \& \quad g(x_2) = x_2^3 - 3x_2$$

$$\Rightarrow x_1^3 - 3x_1 = x_2^3 - 3x_2$$

$$x_1^3 - 3x_1 - x_2^3 + 3x_2 = 0$$

$$x_1^3 - x_2^3 - 3x_1 + 3x_2 = 0$$

$$(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) - 3(x_1 - x_2) = 0$$

$$(x_1 - x_2)[x_1^2 + x_1x_2 + x_2^2 - 3] = 0$$

$$x_1 - x_2 = 0 \quad \wedge \quad x_1^2 + x_1x_2 + x_2^2 - 3 \neq 0$$

$$x_1 = x_2 \quad \Rightarrow \quad x_1 \neq x_2$$

As $x_1 \neq x_2$ therefore $g(x)$ is not injective.

For Surjective $g(x) = x^3 - 3x$

As $g(x)$ is cubic function therefore

$$D_{g(x)} = \mathbb{R} = (-\infty, +\infty)$$

$$R_{g(x)} = \mathbb{R} = (-\infty, +\infty) = \text{Co-domain}$$

Hence $g(x)$ is surjective.