

Chapter No 3

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Theory of Quadratic Functions

Exercise #3.1

Q.1 Find the maximum or minimum value of the following Quadratic functions by Completing Square.

i) $f(x) = x^2 + 6x + 13$

Sol: $f(x) = x^2 + 6x + 13$

$$f(x) = x^2 + 6x + (3)^2 - (3)^2 + 13$$

$$f(x) = (x+3)^2 - 9 + 13$$

$$f(x) = (x+3)^2 + 4 \quad \text{--- (1)}$$

As $f(x) = a(x-h)^2 + k$ --- (2)

from (1) & (2)

$$a = 1, \quad h = -3 \quad \& \quad k = 4$$

Here $a = 1 > 0$ therefore

The min. value of $f(x)$ at

$$x = -3 \quad \text{is} \quad 4$$

ii) $f(x) = x^2 + 4x$

Sol: $f(x) = x^2 + 4x$

$$f(x) = x^2 + 4x + (2)^2 - (2)^2$$

$$f(x) = (x+2)^2 - 4 \quad \text{--- (2)}$$

As $f(x) = a(x-h)^2 + k$ --- (2)

from (1) & (2)

$$a = 1, \quad h = -2 \quad \& \quad k = -4$$

Here $a = 1 > 0$, therefore

The min value of $f(x)$ at

$$x = -2 \quad \text{is} \quad -4$$

iii) $f(x) = -x^2 + 8x + 13$

Sol: $f(x) = -x^2 + 8x + 13$

$$f(x) = -(x^2 - 8x - 13)$$

$$f(x) = -(x^2 - 8x + (4)^2 - (4)^2 - 13)$$

$$f(x) = -[(x-4)^2 - 16 - 13]$$

$$f(x) = -[(x-4)^2 - 29]$$

$$f(x) = -(x-4)^2 + 29$$

Since $a = -1, \quad h = 4 \quad k = 29$

Here $a = -1 < 0$, therefore

The Max. value of $f(x)$ at

$$x = 4 \quad \text{is} \quad 29$$

$$iv) f(x) = -x^2 - 3x - 5$$

$$\text{Sol: } f(x) = -x^2 - 3x - 5$$

$$f(x) = -(x^2 + 3x + 5)$$

$$f(x) = -\left[x^2 + 3x + \left(\frac{3}{2}\right)^2 + 5 - \left(\frac{3}{2}\right)^2\right]$$

$$f(x) = -\left[\left(x + \frac{3}{2}\right)^2 + 5 - \frac{9}{4}\right]$$

$$f(x) = -\left[\left(x + \frac{3}{2}\right)^2 + \frac{20-9}{4}\right]$$

$$f(x) = -\left[\left(x + \frac{3}{2}\right)^2 + \frac{11}{4}\right]$$

$$f(x) = -\left(x + \frac{3}{2}\right)^2 - \frac{11}{4} \quad (1)$$

$$\text{As } f(x) = a(x-h)^2 + k \quad (2)$$

from (1) & (2)

$$a = -1, h = -\frac{3}{2} \text{ & } k = -\frac{11}{4}$$

Here $a = -1 < 0$, therefore

The Max. value of $f(x)$ at

$$x = -\frac{3}{2} \text{ is } -\frac{11}{4}$$

$$v) f(x) = 3x^2 + 6x - 13$$

$$\text{Sol: } f(x) = 3x^2 + 6x - 13$$

$$f(x) = 3(x^2 + 2x - \frac{13}{3})$$

$$f(x) = 3\left[x^2 + 2x + (1)^2 - (1)^2 - \frac{13}{3}\right]$$

$$f(x) = 3\left[(x+1)^2 - 1 - \frac{13}{3}\right]$$

$$f(x) = 3\left[(x+1)^2 + \frac{-3-13}{3}\right]$$

$$f(x) = 3(x+1)^2 - 3 \cdot \frac{16}{3}$$

$$f(x) = 3(x+1)^2 - 16 \quad (1)$$

$$\text{As } f(x) = (x-h)^2 + k \quad (2)$$

from (1) & (2)

$$a = 3, h = -1 \text{ & } k = -16$$

Here $a = 3 > 0$, therefore

The Min. value of $f(x)$ at

$$x = -1 \text{ is } -16$$

$$vi) f(x) = -2x^2 - x + 21$$

$$\text{Sol: } f(x) = -2x^2 - x + 21$$

$$f(x) = -2\left[x^2 + \frac{x}{2} - \frac{21}{2}\right]$$

$$f(x) = -2\left[x^2 + \frac{x}{2} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - \frac{21}{2}\right]$$

$$f(x) = -2\left[\left(x + \frac{1}{4}\right)^2 - \frac{1}{16} - \frac{21}{2}\right]$$

$$f(x) = -2\left[\left(x + \frac{1}{4}\right)^2 + \frac{-1-168}{16}\right]$$

$$f(x) = -2\left[\left(x + \frac{1}{4}\right)^2 - \frac{169}{16}\right]$$

$$f(x) = -2\left(x + \frac{1}{4}\right)^2 + 2 \cdot \frac{169}{16}$$

$$f(x) = -2\left(x + \frac{1}{4}\right)^2 + \frac{169}{8}$$

$$\therefore a = -2, h = \frac{1}{4} \text{ & } k = \frac{169}{8}$$

Here $a = -2 < 0$ therefore

The Max. value of $f(x)$ at

$$x = \frac{1}{4} \text{ is } 169/8$$

Q.2 Find the maximum or minimum point by sketching the following functions. Also find their domain & range.

i) $f(x) = x^2 - 4x$

Sol: $f(x) = x^2 - 4x$

$f(x) = x^2 - 4x + (2)^2 - (2)^2$

$f(x) = (x-2)^2 - 4$ — (1)

As $f(x) = a(x-h)^2 + k$ — (2)

from (1) & (2)

$a=1, h=2$ & $k=-4$

For x-Intercept put $f(x)=0$

$0 = x^2 - 4x$

$x(x-4) = 0$

$x=0$ & $x-4=0$

$x=4$

Points $(0,0)$ & $(4,0)$

Here $a=1 > 0$ therefore the min. value of $f(x)$ at $x=2$ is -4

Since $a > 0$ therefore parabola open upward.

For Domain

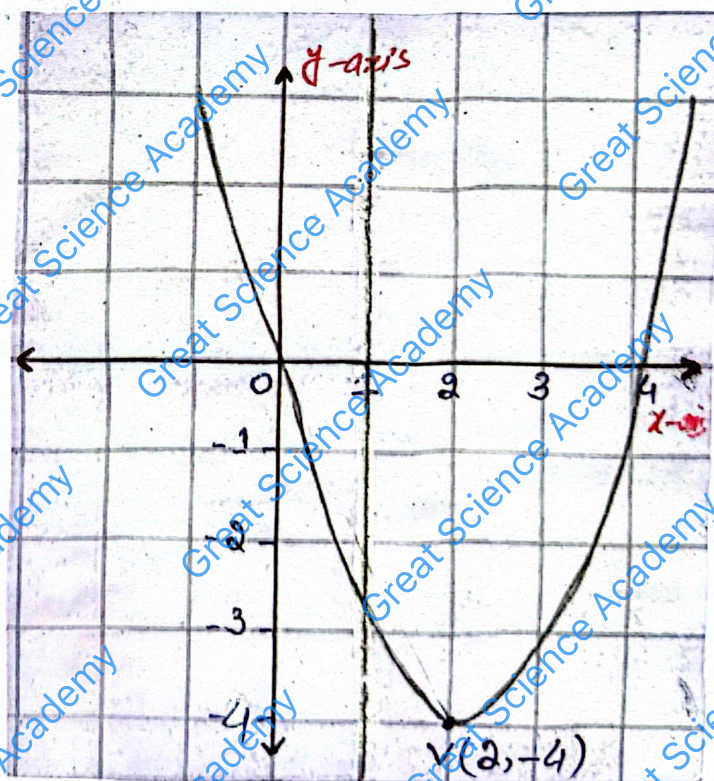
As $f(x)$ is a Quadratic function therefore

$D_f = \mathbb{R}$ or $(-\infty, +\infty)$

For Range

Since Min. value of $f(x)$ is -4 therefore

$R_f = [-4, +\infty)$



$$ii) f(x) = x^2 - 5x + 6$$

$$\text{Sol:- } f(x) = x^2 - 5x + 6$$

$$f(x) = x^2 - 5x + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 6$$

$$f(x) = \left(x - \frac{5}{2}\right)^2 + \frac{6}{1} - \frac{25}{4}$$

$$f(x) = \left(x - \frac{5}{2}\right)^2 + \frac{24 - 25}{4}$$

$$f(x) = \left(x - \frac{5}{2}\right)^2 - \frac{1}{4} \quad (1)$$

$$\text{As } f(x) = a(x-h)^2 + k \quad (2)$$

from (1) & (2)

$$a = 1, \quad h = \frac{5}{2} \quad \& \quad k = -\frac{1}{4}$$

Here $a = 1 > 0$ therefore the min value of $f(x)$ at $x = \frac{5}{2}$ is $-\frac{1}{4}$

Since $a > 0$ therefore parabola open upward.

For Domain

As $f(x)$ is Quadratic function therefore

$$D_f = \mathbb{R} \text{ or } (-\infty, +\infty)$$

For Range

Since the Min. value of $f(x)$ is $-\frac{1}{4}$ therefore

$$R_f = \left[-\frac{1}{4}, +\infty\right)$$

For X-Intercept put $f(x) = 0$

$$x^2 - 5x + 6 = 0$$

$$x^2 - 3x - 2x + 6 = 0$$

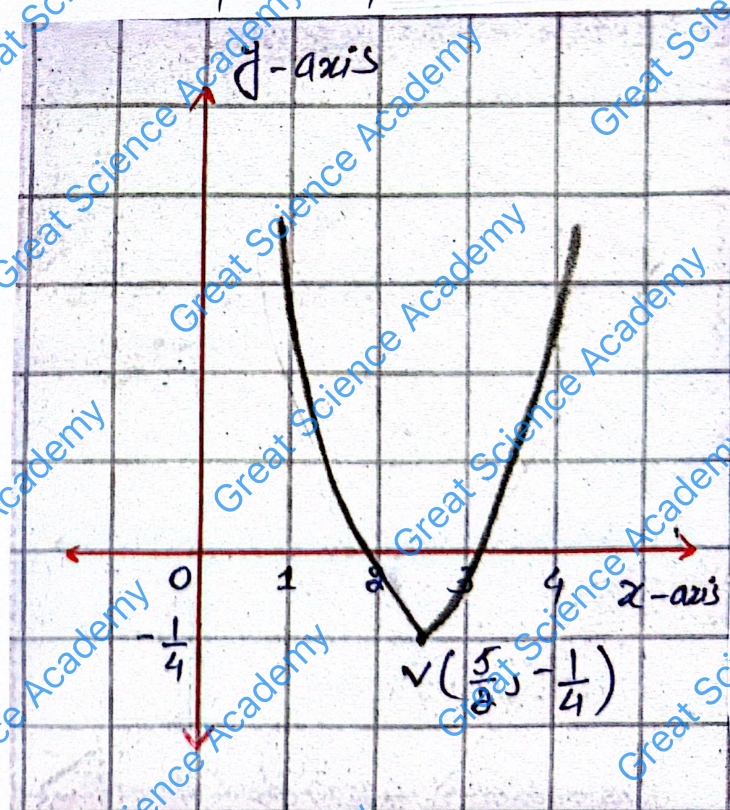
$$x(x-3) - 2(x-3) = 0$$

$$(x-3)(x-2) = 0$$

$$x-3 = 0 \quad \& \quad x-2 = 0$$

$$x = 3 \quad \& \quad x = 2$$

Points $(3, 0)$ & $(2, 0)$



$$\text{iii) } f(x) = -x^2 + 2x - 8$$

$$\text{Sol:- } f(x) = -x^2 + 2x - 8$$

$$f(x) = -(x^2 - 2x + 8)$$

$$f(x) = -(x^2 - 2x + (1)^2 - (1)^2 + 8)$$

$$f(x) = -[(x-1)^2 - 1 + 8]$$

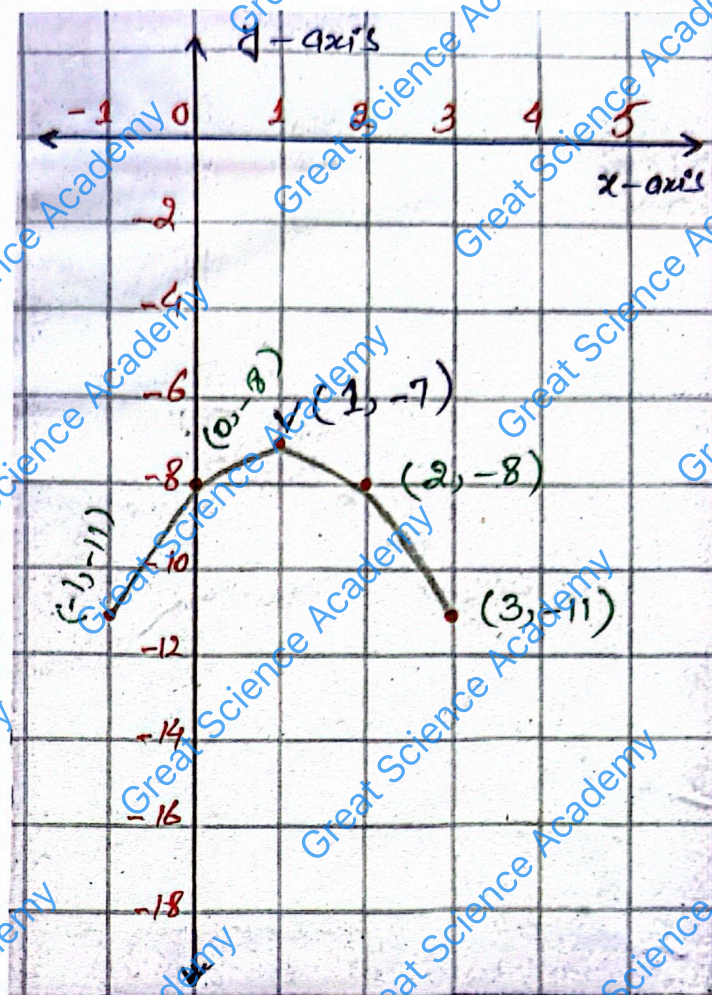
$$f(x) = -[(x-1)^2 + 7]$$

$$f(x) = -(x-1)^2 - 7 \quad \text{--- (1)}$$

$$\text{As } f(x) = a(x-h)^2 + k \quad \text{--- (2)}$$

from (1) & (2)

$$a = -1, h = 1 \text{ \& } k = -7$$



Here $a = -1 < 0$ therefore the Max. value of $f(x)$ at $x = 1$ is -7

Since $a < 0$ therefore parabola open down-ward.

x	-1	0	1	2	3	4	5
$f(x)$	-11	-8	-7	-8	-11	-16	-23

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For Domain

As $f(x)$ is Quadratic function therefore

$$D_f = \mathbb{R} \text{ or } (-\infty, +\infty)$$

For Range

The Max. value of $f(x)$ is -7 therefore

$$R_f = (-\infty, -7]$$

$$iv) f(x) = x^2 - 4x + 4$$

$$\text{Sol:- } f(x) = x^2 - 4x + 4$$

$$f(x) = x^2 - 4x + (2)^2 - (2)^2 + 4$$

$$f(x) = (x-2)^2 - 4 + 4$$

$$f(x) = (x-2)^2 + 0 \quad \text{--- (1)}$$

$$\text{As } f(x) = a(x-h)^2 + k \quad \text{--- (2)}$$

from (1) & (2)

$$a = 1, h = 2 \text{ \& } k = 0$$

Here $a = 1 > 0$, therefore the Min. value of $f(x)$ at $x = 2$ is 0

Since $a > 0$ therefore parabola open upward.

For Domain

As $f(x)$ is Quadratic function therefore

$$D_f = \mathbb{R} \text{ or } (-\infty, +\infty)$$

For Range

Since the Min. value of $f(x)$ is 0, therefore

$$R_f = [0, +\infty)$$

For x-Intercept put $f(x) = 0$

$$x^2 - 4x + 4 = 0$$

$$x^2 - 2(x)(2) + (2)^2 = 0$$

$$(x-2)^2 = 0$$

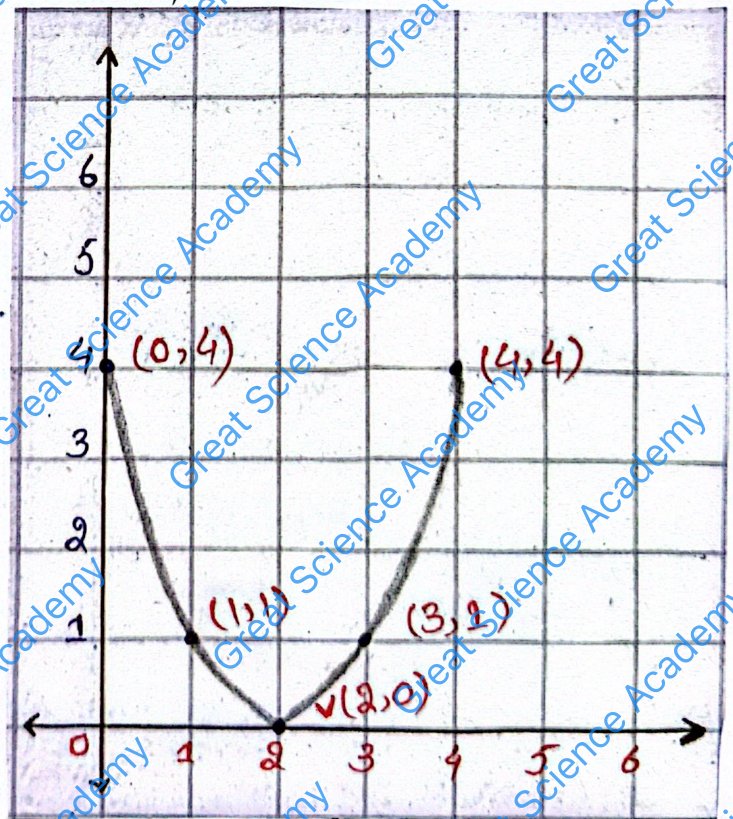
$$\Rightarrow x - 2 = 0 \Rightarrow x = 2$$

1st point (2, 0)

For y-Intercept put $x = 0$

$$f(x) = 0 - 4(0) + 4 = 4$$

2nd point (0, 4)



$$(v) f(x) = x^2 + 2x - 8.3$$

$$\text{Sol: } f(x) = x^2 + 2x - 8.3$$

$$f(x) = x^2 + 2x + (1)^2 - (1)^2 - 8.3$$

$$f(x) = (x+1)^2 - 9.3 \quad \text{--- (1)}$$

$$\text{As } f(x) = a(x-h)^2 + k \quad \text{--- (2)}$$

from (1) & (2)

$$a = 1, \quad h = -1 \quad \& \quad k = -9.3$$

For x-Intercept put $f(x) = 0$

$$x^2 + 2x - 8.3 = 0$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-8.3)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{4 + 33.2}}{2}$$

$$x = \frac{-2 \pm \sqrt{37.2}}{2}$$

$$x = \frac{-2 \pm 6.099}{2} = \frac{-2 \pm 6}{2}$$

$$x = \frac{-2+6}{2} \quad \& \quad x = \frac{-2-6}{2}$$

$$x = 2$$

$$x = -4$$

points are (2, 0), (-4, 0)

Here $a = 1 > 0$, therefore the Min. value of $f(x)$ at $x = -1$ is -9.3

Since $a > 0$ therefore parabola open upward.

For Domain

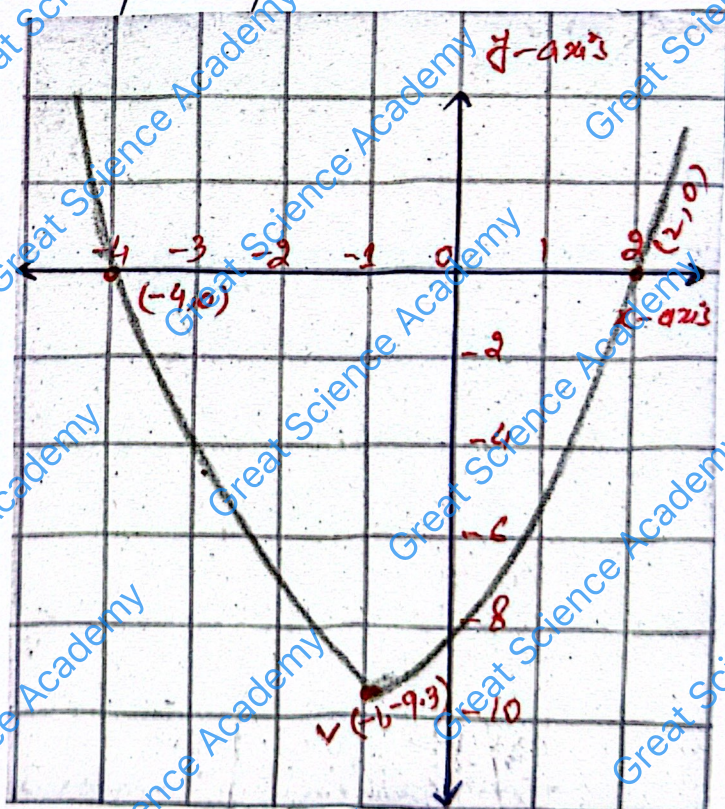
As $f(x)$ is Quadratic function therefore

$$D_f = \mathbb{R} \text{ or } (-\infty, +\infty)$$

For Range

Since the Min. value of $f(x)$ is -9.3 therefore

$$R_f = [-9.3, +\infty)$$



$$vi) f(x) = 6 - x - x^2$$

$$\text{Sol:- } f(x) = 6 - x - x^2$$

$$f(x) = -(x^2 + x - 6)$$

$$f(x) = -\left[x^2 + x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - 6\right]$$

$$f(x) = -\left[\left(x + \frac{1}{2}\right)^2 - 6 - \frac{1}{4}\right]$$

$$f(x) = -\left[\left(x + \frac{1}{2}\right)^2 + \frac{24-1}{4}\right]$$

$$f(x) = -\left[\left(x + \frac{1}{2}\right)^2 - \frac{25}{4}\right]$$

$$f(x) = -\left(x + \frac{1}{2}\right)^2 + \frac{25}{4} \quad (1)$$

$$\Rightarrow a = -1, \quad h = -\frac{1}{2} \quad \& \quad k = \frac{25}{4}$$

Here $a = -1 < 0$, therefore the Max. value of $f(x)$ at $x = -\frac{1}{2}$ is $\frac{25}{4}$

Since $a < 0$ therefore parabola open downward.

For Domain

As $f(x)$ is Quadratic function therefore

$$D_f = \mathbb{R} = (-\infty, +\infty)$$

For Range

Since the Max. value of $f(x)$ is $\frac{25}{4}$ therefore

$$R_f = (-\infty, \frac{25}{4}]$$

For x-intercepts put $f(x) = 0$

$$-x^2 - x + 6 = 0$$

$$\Rightarrow x^2 + x - 6 = 0$$

$$x^2 + 3x - 2x - 6 = 0$$

$$x(x+3) - 2(x+3) = 0$$

$$(x+3)(x-2) = 0$$

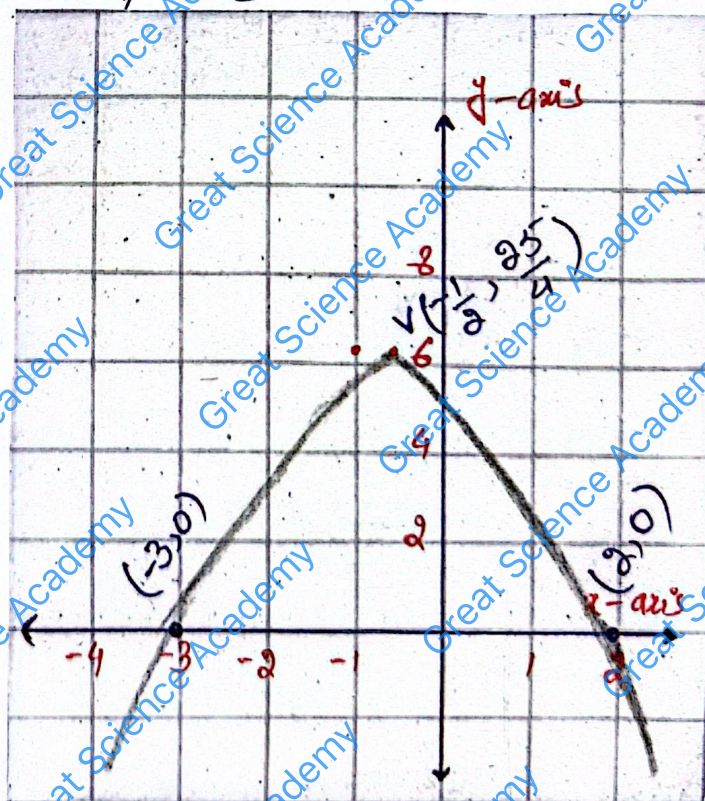
$$x+3 = 0 \quad \wedge \quad x-2 = 0$$

$$x = -3$$

$$x = 2$$

The points are

$$(-3, 0) \quad \& \quad (2, 0)$$



Q.3 Find the inverse of the following quadratic functions.
Also find their Domain and Range:

i) $f(x) = x^2 - 3, x \leq 0$

Sol:- $f(x) = x^2 - 3, x \leq 0$ — (i)

As $f(x) = y$ — (ii)

$\Rightarrow x = f^{-1}(y)$ — (iii)

From (i) & (ii)

$y = x^2 - 3 \Rightarrow y + 3 = x^2$

Taking Square root

$\sqrt{x^2} = \sqrt{y+3}$

$\Rightarrow x = \pm \sqrt{y+3}$

$\because x \leq 0$ therefore

$x = -\sqrt{y+3}$ put in (iii)

$-\sqrt{y+3} = f^{-1}(y)$

Replace y by x

$\Rightarrow \boxed{f^{-1}(x) = -\sqrt{x+3}}$

For Domain of $f(x)$

$\boxed{D_f = (-\infty, 0]}$

For Range of $f(x)$

$\boxed{R_f = [-3, +\infty)}$

For Domain of $f^{-1}(x)$

$\boxed{D_{f^{-1}} = [-3, +\infty)}$

For Range of $f^{-1}(x)$

$\boxed{R_{f^{-1}} = (-\infty, 0]}$

ii) $f(x) = x^2 + 6x + 4, x \leq -3$

Sol $f(x) = x^2 + 6x + 4, x \leq -3$

$f(x) = x^2 + 6x + (3)^2 - (3)^2 + 4$

$f(x) = (x+3)^2 - 9 + 4$

$f(x) = (x+3)^2 - 5$ — (1)

As $f(x) = a(x-h)^2 + k$ — (2)

from (1) & (2)

$a = 1, h = -3 \text{ \& } k = -5$

Here $a = 1 > 0$, therefore

The Min. value of $f(x)$ at $x = -3$ is -5

Since $a > 0$ therefore the Parabola open upward

Domain of $f(x)$

$$D_f = (-\infty, -3]$$

Range of $f(x)$

Since the Min. value of $f(x)$ is -5 therefore

$$R_f = (-5, +\infty)$$

$$f(x) = (x+3)^2 - 5 \quad \text{--- (i)}$$

$$\text{As } y = f(x) \quad \text{--- (ii)}$$

$$\Rightarrow f^{-1}(y) = x \quad \text{--- (iii)}$$

from (i) & (ii)

$$y = (x+3)^2 - 5$$

$$y+5 = (x+3)^2$$

Taking square root

$$\Rightarrow \pm \sqrt{y+5} = (x+3)$$

$\therefore x < -3$ therefore

$$x+3 = -\sqrt{y+5}$$

$$x = -\sqrt{y+5} - 3$$

put in (iii) we get

$$f^{-1}(y) = -\sqrt{y+5} - 3$$

Replace y by x

$$f^{-1}(x) = -\sqrt{x+5} - 3$$

Domain of $f^{-1}(x)$

$$D_{f^{-1}} = (-5, +\infty)$$

Range of $f^{-1}(x)$

$$R_{f^{-1}} = (-\infty, -3]$$

$$\text{iii) } f(x) = 2x^2 - 8x + 11, x \geq 2$$

$$\text{Sol: } f(x) = 2x^2 - 8x + 11; x \geq 2$$

$$f(x) = 2\left(x^2 - 4x + \frac{11}{2}\right)$$

$$f(x) = 2\left(x^2 - 4x + (2)^2 - (2)^2 + \frac{11}{2}\right)$$

$$f(x) = 2\left[(x-2)^2 + \frac{11}{2} - 4\right]$$

$$f(x) = 2\left[(x-2)^2 + \frac{11-8}{2}\right]$$

$$f(x) = 2(x-2)^2 + 3 \quad \text{--- (2)}$$

$$\text{As } f(x) = a(x-h)^2 + k$$

$$a=2 \quad h=2 \quad \& \quad k=3$$

Here $a=2 > 0$ therefore

the Min. value of $f(x)$

at $x=2$ is 3

Since $a > 0$ therefore parabola open upward.

Domain of $f(x)$

$$D_{f(x)} = [2, +\infty)$$

Range of $f(x)$

Since the min. value of $f(x)$ is 3 therefore

$$R_{f(x)} = [3, +\infty)$$

Now $f(x) = 2(x-2)^2 + 3$ — (i)

Let $y = f(x)$ — (ii)

$\Rightarrow f^{-1}(y) = x$ — (iii)

from (i) & (ii)

$$y = 2(x-2)^2 + 3$$

$$\Rightarrow (x-2)^2 = \frac{y-3}{2}$$

Taking square root

$$\Rightarrow x-2 = \pm \sqrt{\frac{y-3}{2}}$$

$\therefore x \geq 2$ therefore

$$x-2 = \sqrt{\frac{y-3}{2}}$$

$$\Rightarrow x = \sqrt{\frac{y-3}{2}} + 2$$

putting the value of x in (iii)

$$f^{-1}(y) = \sqrt{\frac{y-3}{2}} + 2$$

Replace y by x

$$f^{-1}(x) = \sqrt{\frac{x-3}{2}} + 2$$

Domain of $f^{-1}(x)$

$$D_{f^{-1}(x)} = [3, +\infty)$$

Range of $f^{-1}(x)$

$$R_{f^{-1}(x)} = [2, +\infty)$$

iv) $f(x) = 3x^2 - 2x + 6$ $x \geq 5$

$$f(x) = 3x^2 - 2x + 6 ; x \geq 5$$

$$f(x) = 3\left(x^2 - \frac{2}{3}x + 2\right)$$

$$f(x) = 3\left(x^2 - \frac{2}{3}x + \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2 + 2\right)$$

$$f(x) = 3\left[\left(x - \frac{1}{3}\right)^2 + 2 - \frac{1}{9}\right]$$

$$f(x) = 3\left[\left(x - \frac{1}{3}\right)^2 + \frac{18-1}{9}\right]$$

$$f(x) = 3\left(x - \frac{1}{3}\right)^2 + 3 \times \frac{17}{3 \times 3}$$

$$f(x) = 3\left(x - \frac{1}{3}\right)^2 + \frac{17}{3} \quad \text{--- (i)}$$

$$\text{As } f(x) = a(x-h)^2 + k \quad \text{--- (ii)}$$

from (i) & (ii)

$$a = 3, h = \frac{1}{3}, k = \frac{17}{3}$$

Here $a = 3 > 0$ therefore

the Min. value of $f(x)$

$$\text{at } x = \frac{1}{3} \text{ is } \frac{17}{3}$$

Here $a > 0$ therefore the

Parabola open upward.

Domain of $f(x)$

$$D_{f(x)} = [5, +\infty)$$

Range of $f(x)$

$$\text{Range} = [7, +\infty)$$

$$\text{Now } f(x) = 3\left(x - \frac{1}{3}\right)^2 + \frac{17}{3} \quad \text{--- (i)}$$

$$\text{Let } y = f(x) \quad \text{--- (ii)}$$

$$\Rightarrow f^{-1}(y) = x \quad \text{--- (iii)}$$

from (i) & (ii)

$$y = 3\left(x - \frac{1}{3}\right)^2 + \frac{17}{3}$$

$$y - \frac{17}{3} = 3\left(x - \frac{1}{3}\right)^2$$

$$\frac{3y - 17}{3} = 3\left(x - \frac{1}{3}\right)^2$$

$$\Rightarrow \left(x - \frac{1}{3}\right)^2 = \frac{3y - 17}{9}$$

Taking square root

$$\Rightarrow x - \frac{1}{3} = \pm \sqrt{\frac{3y - 17}{9}}$$

$\because x > 5$ therefore

$$x - \frac{1}{3} = \sqrt{\frac{3y - 17}{9}}$$

$$x = \frac{1}{3} + \frac{\sqrt{3y - 17}}{3}$$

$$x = \frac{1 + \sqrt{3y - 17}}{3} \quad \text{put in (iii)}$$

$$f^{-1}(y) = \frac{1 + \sqrt{3y - 17}}{3}$$

Replace y by x

$$f^{-1}(x) = \frac{1 + \sqrt{3x - 17}}{3}$$

Domain of $f^{-1}(x)$

$$D_{f^{-1}(x)} = [7, +\infty)$$

Range of $f^{-1}(x)$

$$R_{f^{-1}(x)} = [5, +\infty)$$

$$y = f(x) = 2(x-3)^2 + 1, x \geq 3$$

$$\text{Sol: } f(x) = 2(x-3)^2 + 1 \quad \text{--- (1)}$$

$$\text{A: } f(x) = a(x-h)^2 + k \quad \text{--- (2)}$$

from (1) & (2)

$$a = 2, \quad h = 3 \quad \& \quad k = 1$$

Here $a = 2 > 0$, therefore the

Min value of $f(x)$ at $x = 3$ is 1

Since $a > 0$, therefore the Parabola
open upward

Domain of $f(x)$

$$D_{f(x)} = [3, +\infty)$$

Range of $f(x)$

Since the Min value of
 $f(x)$ is 1 therefore

$$R_{f(x)} = [1, +\infty)$$

$$\text{Let } f(x) = y \quad \text{--- (i)}$$

$$\Rightarrow x = f^{-1}(y) \quad \text{--- (3)}$$

from (1) & (3)

$$y = 2(x-3)^2 + 1$$

$$y - 1 = 2(x-3)^2$$

$$\Rightarrow (x-3)^2 = \frac{y-1}{2}$$

Taking square root

$$\Rightarrow x - 3 = \pm \sqrt{\frac{y-1}{2}}$$

$\because x \geq 3$ therefore

$$x - 3 = \sqrt{\frac{y-1}{2}}$$

$$x = 3 + \sqrt{\frac{y-1}{2}} \quad \text{put in (3)}$$

$$f^{-1}(y) = 3 + \sqrt{\frac{y-1}{2}}$$

Replace y by x

$$\Rightarrow f^{-1}(x) = 3 + \sqrt{\frac{x-1}{2}}$$

Domain of $f^{-1}(x)$

$$D_{f^{-1}(x)} = [1, +\infty)$$

Range of $f^{-1}(x)$

$$R_{f^{-1}(x)} = [3, +\infty)$$

$$vi) f(x) = -3(x+4)^2 - 5, x < -4$$

$$Sol:- f(x) = -3(x+4)^2 - 5 \quad (1)$$

$$As f(x) = a(x-h)^2 + k \quad (2)$$

From (1) & (2)

$$a = -3, h = -4 \text{ \& } k = -5$$

Here $a = -3 < 0$, therefore

The Max. value of $f(x)$

at $x = -4$ is -5

Since $a < 0$, therefore the
Parabola open down-ward.

Domain of $f(x)$

$$D_{f(x)} = (-\infty, -4)$$

Range of $f(x)$

$$R_{f(x)} = (-\infty, -5)$$

Due the Max value of

$f(x)$ is -5

$$Let y = f(x) \quad (3)$$

$$\Rightarrow f^{-1}(y) = x \quad (4)$$

From (2) & (3)

$$y = -3(x+4)^2 - 5$$

$$y + 5 = -3(x+4)^2$$

$$\Rightarrow (x+4)^2 = \frac{-(y+5)}{3}$$

Taking square root

$$\Rightarrow x+4 = \pm \sqrt{\frac{-(y+5)}{3}}$$

Since $x < -4$ therefore

$$x+4 = -\sqrt{\frac{-(y+5)}{3}}$$

$$x = -4 - \sqrt{\frac{-(y+5)}{3}}$$

put in eq. (4)

$$f^{-1}(y) = -4 - \sqrt{\frac{-(y+5)}{3}}$$

Replace y by x

$$f^{-1}(x) = -4 - \sqrt{\frac{-(x+5)}{3}}$$

Domain of $f^{-1}(x)$

$$D_{f^{-1}(x)} = (-\infty, -5)$$

Range of $f^{-1}(x)$

$$R_{f^{-1}(x)} = (-\infty, -4)$$

4. Solve the following absolute value quadratic equations and inequalities:

$$(i) |x^2 + 1| = 5$$

$$\text{Sol: } |x^2 + 1| = 5 \Rightarrow x^2 + 1 = \pm 5$$

$$x^2 + 1 = 5$$

$$x^2 = 5 - 1$$

$$x^2 = 4$$

Taking sq. root.

$$\Rightarrow \boxed{x = \pm 2}$$

$$x^2 + 1 = -5$$

$$x^2 = -5 - 1$$

$$x^2 = -6$$

Taking sq. root

$$\Rightarrow x = \pm \sqrt{-6} \notin \mathbb{Z} \text{ (Not int)}$$

Checking

$$\text{at } x = -2 \Rightarrow |(-2)^2 + 1| = 5 \Rightarrow 5 = 5 \text{ (True)}$$

$$\text{at } x = 2 \Rightarrow |(2)^2 + 1| = 5 \Rightarrow 5 = 5 \text{ (True)}$$

$$\boxed{\text{Ans} = \{ \pm 2 \}}$$

Ans

$$ii) |x^2 + 5x + 4| = 0$$

$$\text{Sol: } |x^2 + 5x + 4| = 0 \Rightarrow x^2 + 5x + 4 = 0$$

$$x^2 + 4x + x + 4 = 0 \Rightarrow x(x+4) + 1(x+4) = 0$$

$$(x+1)(x+4) = 0 \Rightarrow x+1 = 0 \quad \& \quad x+4 = 0$$

$$\boxed{x = -1}$$

$$\boxed{x = -4}$$

$$\text{at } x = -4 \Rightarrow |(-4)^2 + 5(-4) + 4| = 0 \Rightarrow 0 = 0 \text{ (True)}$$

$$\text{at } x = -1 \Rightarrow |(-1)^2 + 5(-1) + 4| = 0 \Rightarrow 0 = 0 \text{ (True)}$$

$$\boxed{S = \{-4, -1\}}$$

Ans

$$\text{iii) } |x^2 - 6x + 8| = 4$$

$$\text{Sol:- } |x^2 - 6x + 8| = 4 \Rightarrow x^2 - 6x + 8 = \pm 4$$

$$x^2 - 6x + 8 = 4$$

$$x^2 - 6x + 8 - 4 = 0$$

$$x^2 - 6x + 4 = 0$$

using Q. F

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(4)}}{2}$$

$$x = \frac{6 \pm \sqrt{36 - 16}}{2}$$

$$x = \frac{6 \pm \sqrt{20}}{2}$$

$$x = \frac{6 \pm 2\sqrt{5}}{2}$$

$$x = \frac{2(3 \pm \sqrt{5})}{2}$$

$$\boxed{x = 3 \pm \sqrt{5}}$$

$$x^2 - 6x + 8 = -4$$

$$x^2 - 6x + 8 + 4 = 0$$

$$x^2 - 6x + 12 = 0$$

using Q. F

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(12)}}{2}$$

$$x = \frac{6 \pm \sqrt{36 - 48}}{2}$$

$$x = \frac{6 \pm \sqrt{-12}}{2}$$

$$x = \frac{6 \pm 2\sqrt{3}i}{2}$$

$$x = \frac{2(3 \pm \sqrt{3}i)}{2}$$

$$x = 3 \pm \sqrt{3}i \in \mathbb{C} \text{ (Neglect)}$$

Checking

$$\text{or } x = 3 + \sqrt{5} \Rightarrow |(3 + \sqrt{5})^2 - 6(3 + \sqrt{5}) + 8| = 4$$

$$|9 + 5 + 6\sqrt{5} - 18 - 6\sqrt{5} + 8| = 4$$

$$|22 - 18| = 4 \Rightarrow 4 = 4 \text{ (True)}$$

$$\text{or } x = 3 - \sqrt{5} \Rightarrow |(3 - \sqrt{5})^2 - 6(3 - \sqrt{5}) + 8| = 4$$

$$|9 + 5 - 6\sqrt{5} - 18 + 6\sqrt{5} + 8| = 4$$

$$|22 - 18| = 4 \Rightarrow 4 = 4 \text{ (True)}$$

$$\boxed{x = 3 \pm \sqrt{5}}$$

Ans

$$(iv) |3x^2 - 7x + 2| = x^2 - x + 1$$

$$\text{Sol:- } |3x^2 - 7x + 2| = x^2 - x + 1 \Rightarrow 3x^2 - 7x + 2 = \pm(x^2 - x + 1)$$

$$3x^2 - 7x + 2 = x^2 - x + 1$$

$$3x^2 - 7x + 2 - x^2 + x - 1 = 0$$

$$2x^2 - 6x + 1 = 0$$

Using Q.F

$$x = \frac{6 \pm \sqrt{36 - 8}}{2(2)}$$

$$x = \frac{6 \pm \sqrt{28}}{2(2)}$$

$$3x^2 - 7x + 2 = -(x^2 - x + 1)$$

$$3x^2 - 7x + 2 + x^2 - x + 1 = 0$$

$$4x^2 - 8x + 3 = 0$$

$$4x^2 - 6x - 2x + 3 = 0$$

$$2x(2x - 3) - 1(2x - 3) = 0$$

$$(2x - 3)(2x - 1) = 0$$

$$2x - 3 = 0 \quad \vee \quad 2x - 1 = 0$$

$$\Rightarrow \boxed{x = 3/2}$$

$$\boxed{x = 1/2}$$

$$x = \frac{6 \pm 2\sqrt{7}}{4} = \frac{2(3 \pm \sqrt{7})}{4} \Rightarrow$$

$$x = \frac{3 \pm \sqrt{7}}{2}$$

Checking

$$\text{or } x = \frac{3}{2} \Rightarrow \left| 3\left(\frac{3}{2}\right)^2 - 7\left(\frac{3}{2}\right) + 2 \right| = \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right) + 1$$

$$\left| \frac{27}{4} - \frac{21}{2} + 2 \right| = \frac{9}{4} - \frac{3}{2} + 1$$

$$\left| \frac{27 - 42 + 8}{4} \right| = \frac{9 - 6 + 4}{4}$$

$$\left| -\frac{7}{4} \right| = \frac{7}{4} \Rightarrow \frac{7}{4} = \frac{7}{4} \text{ (True)}$$

$$\text{or } x = \frac{1}{2} \Rightarrow \left| 3\left(\frac{1}{2}\right)^2 - 7\left(\frac{1}{2}\right) + 2 \right| = \left(\frac{1}{2}\right)^2 - \frac{1}{2} + 1$$

$$\left| \frac{3}{4} - \frac{7}{2} + 2 \right| = \frac{1}{4} - \frac{1}{2} + 1$$

$$\left| \frac{3 - 14 + 8}{4} \right| = \frac{1 - 2 + 4}{4}$$

$$\left| -\frac{3}{4} \right| = \frac{3}{4} \Rightarrow \frac{3}{4} = \frac{3}{4} \text{ (True)}$$

$$\text{or } x = \frac{3 \pm \sqrt{7}}{2} \Rightarrow \left| 3\left(\frac{3 \pm \sqrt{7}}{2}\right)^2 - 7\left(\frac{3 \pm \sqrt{7}}{2}\right) + 2 \right| = \left(\frac{3 \pm \sqrt{7}}{2}\right)^2 - \left(\frac{3 \pm \sqrt{7}}{2}\right) + 1$$

$$\Rightarrow \left| 3\left(\frac{9 + 7 \mp 6\sqrt{7}}{4}\right) - \frac{(21 \pm 7\sqrt{7})}{2} + 2 \right| = \frac{9 + 7 \mp 6\sqrt{7}}{4} - \left(\frac{3 \pm \sqrt{7}}{2}\right) + 1$$

$$\Rightarrow \left| \frac{48 \mp 18\sqrt{7}}{4} - \frac{(21 \pm 7\sqrt{7})}{2} + 2 \right| = \frac{16 \mp 6\sqrt{7}}{4} - \left(\frac{3 \pm \sqrt{7}}{2}\right) + 1$$

$$\left| \frac{48 \mp 18\sqrt{7} - 42 \mp 14\sqrt{7} + 8}{4} \right| = \frac{16 \mp 6\sqrt{7} - 6 \mp 2\sqrt{7} + 4}{4}$$

$$\left| \frac{16 \pm 32\sqrt{7}}{4} \right| = \frac{16 \pm 32\sqrt{7}}{4} \Rightarrow 4 \frac{(4 \pm 8\sqrt{7})}{4} = 4 \frac{(4 \pm 8\sqrt{7})}{4}$$

$$\Rightarrow 4 \pm 8\sqrt{7} = 4 \pm 8\sqrt{7} \quad (\text{True})$$

$$S.S = \left\{ \frac{3 \pm \sqrt{7}}{2}, \frac{3}{2}, \frac{1}{2} \right\}$$

Ans

(v) $|x^2 - 4| < 5$

Sol:- $|x^2 - 4| < 5 \Rightarrow -5 < x^2 - 4 < 5$

$$-5 < x^2 - 4$$

$$-5 + 4 < x^2$$

$$-1 < x^2$$

Taking sq. root

$$\Rightarrow 1 < x \quad (\text{Neglect})$$

$$x^2 - 4 < 5$$

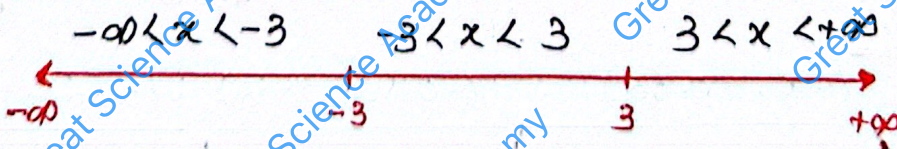
$$x^2 < 5 + 4$$

$$x^2 < 9$$

Taking sq. root

$$x < \pm 3 \Rightarrow -3 < x < 3$$

Real Line



There are three sub-intervals

at $x = -4 \Rightarrow$

$$(-4)^2 - 4 < 5 \Rightarrow 12 < 5 \quad (\text{False})$$

at $x = 2 \Rightarrow$

$$(2)^2 - 4 < 5 \Rightarrow 0 < 5 \quad (\text{True})$$

at $x=6 \Rightarrow (6)^2 - 4 < 5 \Rightarrow 31 < 5$ (False)

$$S = \{(-3, 3)\}$$

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vi) $|x^2 - 3x + 2| > 4$

Sol:- $|x^2 - 3x + 2| > 4 \Rightarrow -4 > x^2 - 3x + 2 > 4$

$$-4 > x^2 - 3x + 2$$

$$0 > x^2 - 3x + 2 + 4$$

$$0 > x^2 - 3x + 6$$

$$\Rightarrow x^2 - 3x + 6 = 0$$

using Q.F

$$x = \frac{3 \pm \sqrt{9-24}}{2}$$

$$x = \frac{3 \pm \sqrt{-15}}{2}$$

$$x = \frac{3 \pm \sqrt{15}i}{2} \in \mathbb{Z} \text{ (Neglect)}$$

Real Line

$$x^2 - 3x + 2 > 4$$

$$x^2 - 3x + 2 - 4 > 0$$

$$x^2 - 3x - 2 > 0$$

$$\Rightarrow x^2 - 3x - 2 = 0$$

using Q.F

$$x = \frac{3 \pm \sqrt{9+17}}{2}$$

$$x = \frac{3 \pm \sqrt{17}}{2}$$

$$x = \frac{3 - \sqrt{17}}{2} \quad \text{and} \quad x = \frac{3 + \sqrt{17}}{2}$$

$$x = -0.56$$

$$x = 3.56$$

Checking

at $x = -2 \Rightarrow (-2)^2 - 3(-2) - 2 > 0 \Rightarrow 8 > 0$ (True)

at $x = 1 \Rightarrow (1)^2 - 3(1) - 2 > 0 \Rightarrow -4 > 0$ (False)

$$\text{at } x=4 \Rightarrow (4)^2 - 3(4) - 2 > 0 \Rightarrow 2 > 0 \text{ (True)}$$

$$\text{S.S} = \left\{ (-\infty, \frac{3-\sqrt{17}}{2}) \right\} \cup \left\{ \left(\frac{3+\sqrt{17}}{2}, +\infty \right) \right\}$$

Ans

$$\text{vii) } |x^2 - 5x + 6| \leq x + 2$$

$$\text{Sol:- } |x^2 - 5x + 6| \leq x + 2 \Rightarrow -(x+2) \leq x^2 - 5x + 6 \leq x+2$$

$$x+2 \geq x^2 - 5x + 6$$

$$0 \geq x^2 - 5x + 6 - x - 2$$

$$0 \geq x^2 - 6x + 4$$

$$\Rightarrow x^2 - 6x + 4 = 0$$

using Q.F

$$x = \frac{6 \pm \sqrt{36-16}}{2}$$

$$x = \frac{6 \pm \sqrt{20}}{2}$$

$$x = \frac{6 \pm 2\sqrt{5}}{2}$$

$$\Rightarrow x = 3 \pm \sqrt{5}$$

$$x = 3 + \sqrt{5} \quad \& \quad x = 3 - \sqrt{5}$$

$$x = 5.2$$

$$x = 0.76$$

$$x^2 - 5x + 6 \geq -(x+2)$$

$$x^2 - 5x + 6 + x + 2 \geq 0$$

$$x^2 - 4x + 8 \geq 0$$

$$\Rightarrow x^2 - 4x + 8 = 0$$

using Q.F

$$x = \frac{4 \pm \sqrt{16-32}}{2}$$

$$x = \frac{4 \pm \sqrt{-16}}{2}$$

$$x = \frac{4 \pm 4i}{2}$$

$$x = 2 \pm 2i \in \mathbb{C}$$

(Neglect)

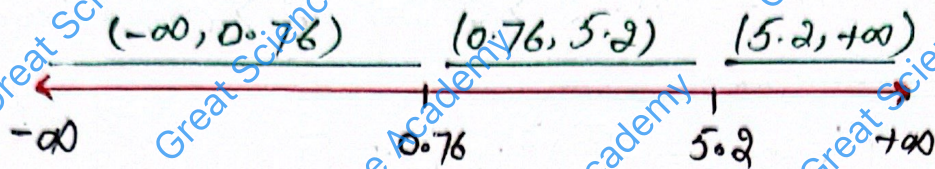
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Real Line



at $x = -1 \Rightarrow (-1)^2 - 6(-1) + 4 \leq 0 \Rightarrow 11 \leq 0$ (False)

at $x = 1 \Rightarrow (1)^2 - 6(1) + 4 \leq 0 \Rightarrow -1 \leq 0$ (True)

at $x = 6 \Rightarrow (6)^2 - 6(6) + 4 \leq 0 \Rightarrow 4 \leq 0$ (False)

$$S = \{ [3 - \sqrt{5}, 3 + \sqrt{5}] \}$$

Ans

vi) $|2x^2 - 3x - 5| < 4$

So:- $|2x^2 - 3x - 5| < 4 \Rightarrow -4 < 2x^2 - 3x - 5 < 4$

$$-4 < 2x^2 - 3x - 5$$

$$0 < 2x^2 - 3x - 5 + 4$$

$$0 < 2x^2 - 3x - 1$$

$$\Rightarrow 2x^2 - 3x - 1 = 0$$

using Q.F

$$x = \frac{3 \pm \sqrt{9+8}}{4}$$

$$x = \frac{3 \pm \sqrt{17}}{4}$$

$$2x^2 - 3x - 5 < 4$$

$$2x^2 - 3x - 5 - 4 < 0$$

$$2x^2 - 3x - 9 < 0$$

$$\Rightarrow 2x^2 - 3x - 9 = 0$$

$$2x^2 - 6x + 3x - 9 = 0$$

$$2x(x-3) + 3(x-3) = 0$$

$$(x-3)(2x+3) = 0$$

$$x-3 = 0 \quad \vee \quad 2x+3 = 0$$

$$\Rightarrow x = 3 \quad x = -\frac{3}{2}$$

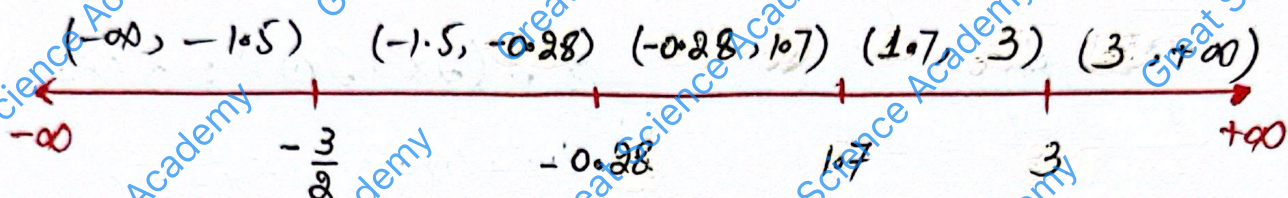
$$x = \frac{3 - \sqrt{17}}{4}$$

$$x = -0.28$$

$$x = \frac{3 + \sqrt{17}}{4}$$

$$x = 1.7$$

Real Line



$$\text{at } x = -2 \Rightarrow |2(-2)^2 - 3(-2) - 5| < 4 \Rightarrow 9 < 4 \text{ (False)}$$

$$\text{at } x = -1.4 \Rightarrow |2(-1.4)^2 - 3(-1.4) - 5| < 4 \Rightarrow 3.42 < 4 \text{ (True)}$$

$$\text{at } x = 1 \Rightarrow |2(1)^2 - 3(1) - 5| < 4 \Rightarrow 6 < 4 \text{ (False)}$$

$$\text{at } x = 2 \Rightarrow |2(2)^2 - 3(2) - 5| < 4 \Rightarrow 1 < 4 \text{ (True)}$$

$$\text{at } x = 4 \Rightarrow |2(4)^2 - 3(4) - 5| < 4 \Rightarrow 15 < 4 \text{ (False)}$$

$$S.S = \left\{ \left(-\frac{3}{2}, \frac{3 - \sqrt{17}}{4} \right) \cup \left(\frac{3 + \sqrt{17}}{4}, 3 \right) \right\}$$

Ans

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